Semilinear approximations of quasilinear parabolic equations

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Preface.

Solvability of quasilinear parabolic equations, that is, equations with highest order elliptic operator dependent on the solution, attracts attention since 1960s. It was considered in classical literature, in particular, by A. Friedman (1964), O.A. Ladyzenskaja, V.A. Solonnikov and N.N. Ural'ceva (1967), D. Gilbarg and N.S. Trudinger (1983) in the elliptic case, A. Lunardi (1984), H. Amann (1986, 1995), A. Yagi (2010) and many others. Probably the most complete studies of the global in time solvability of second order quasilinear parabolic equations can be found in the classical monograph [L-S-U]. Variety of possible solutions to such equations considered in Sobolev and Hölder spaces together with a number of estimates can be found in this early reference.

It is known from the inspiring studies of H. Tanabe, H. Komatsu, T. Kato and P.E. Sobolevskii in the 1960s, described briefly in the last chapter of K. Yosida's monograph, how to treat a general inhomogeneous Cauchy problem

$$\begin{aligned} &u'(t) = A(t)u(t) + F(t), \ t > 0, \\ &u(0) = u_0 \in X, \end{aligned}$$

in a complex Banach space X. A sufficient condition for that was formulated in [TA] using the theory of analytic (holomorphic) semigroups; see [Yo, Chapter XIV.5].

A local existence and uniqueness result of T. Kato [Ka] provides us a tool to treat a Cauchy problem also for a *quasilinear* equation

$$u'(t) = A(t, u(t))u(t) + f(t, u(t)), \ t > 0,$$

constructing its solution as a limit of a sequence $\{u_l\}_l$ of solutions to the time dependent *linear* equations defined inductively as

$$\begin{aligned} &u'_{l+1}(t) = A(t, u_l(t))u_{l+1}(t) + f(t, u_l(t)), \ t > 0, \\ &u_{l+1}(0) = u_0. \end{aligned}$$

In recent paper [Cz-D] we have used the semigroup technique of [HE, Ch-D] to study more general examples within this approach. In fact, similar methods can be applied to solve certain quasilinear Cauchy problems of the form

$$\begin{aligned} &u_t = A(u)u + F(u),\ t > 0, \\ &u(0) = u_0. \end{aligned} \tag{1}$$

We first solve its *viscous semilinear regularizations* with positive parameters ϵ, δ:

$$\begin{aligned} &u^\epsilon_t + \epsilon(-\Delta)^{\alpha+\delta}u^\epsilon = A(u^\epsilon)u^\epsilon + F(u^\epsilon),\ t > 0, \\ &u^\epsilon(0) = u_0, \end{aligned} \tag{2}$$

where $A(u^\epsilon)u^\epsilon + F(u^\epsilon)$ is treated as a nonlinear perturbation of the linear main part $\epsilon(-\Delta)^{\alpha+\delta}u^\epsilon$, then using the uniform in parameter $\epsilon > 0$ a priori estimates of u^ϵ, we let $\epsilon \to 0^+$ in (2) obtaining a weak solution of the original problem (1) in the limit.

References 1.

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The semigroup approach.

We will look at solutions to autonomous partial differential equations within the theory of semigroups. The dynamics generated by such equation, with initial data in a function space E, can be described by the *solution semigroup*

$$T(t) : u(0) \to u(t),$$

that acts in the space E.

Definition

Let E be a metric space. A one parameter family $\{T(t)\}$ of maps $T(t) : E \to E$, $t \geq 0$, is called a C^0-semigroup if

- T(0) is the identity map on E;
- $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$;
- the function

$$[0, \infty) \times E \ni (t, x) \to T(t)x \in E$$

is continuous at each point $(t, x) \in [0, \infty) \times E$.

Semilinear equation with sectorial positive operator.

Following Dan Henry, our starting point are abstract semilinear Cauchy problems with sectorial positive operator A:

$$\begin{aligned} &u_t + Au = F(u), \quad t > 0, \\ &u(0) = u_0. \end{aligned} \tag{3}$$

(compare [HE]). This category includes many important physical equations, like; semilinear heat equation, Navier-Stokes equation, subcritical Quasi-geostrophic equation. But we intent to study also limits of such problems, certain *quasilinear equations*.
Let $S_{a,\phi}$, $a \in R$ and $\phi \in (0, \frac{\pi}{2})$, be a *sector* of the complex plane

$$S_{a,\phi} = \{\lambda \in C : \phi \leq |arg(\lambda - a)| \leq \pi, \lambda \neq a\}. \tag{4}$$

Recall, that a linear, closed and densely defined operator $X \supset D(A) \to X$ in a Banach space X is called *sectorial operator* if there exist a, ϕ as above and $M > 0$ such that:

- the resolvent set $\rho(A)$ of A contains the sector $S_{a,\phi}$,
- $\|(\lambda I - A)^{-1}\|_{\mathcal{L}(X,X)} \leq \frac{M}{|\lambda - a|}$, for each $\lambda \in S_{a,\phi}$.

It is a familiar fact, that for self-adjoint operators in H which are non-negative ($m = 0$), a *square root operator* can be defined. This property can be extend to arbitrary *positive* (or even *non-negative*; [M-S]) sectorial operators in a Banach space. The definition was introduced by A.V. Balakrishnan in 1959/60 and studied in a series of five papers by Hikosaburo Komatsu (e.g. [Ko]).

Recall then the Balakrishnan definition of fractional power of non-negative operator; Let A be a closed linear densely defined operator in a Banach space X, such that its resolvent set contains $(-\infty, 0)$ and the resolvent satisfies:

$$\|\lambda(\lambda + A)^{-1}\| \leq M, \ \lambda > 0. \tag{5}$$

Then, for $\eta \in (0, 1), \phi \in D(A)$,

$$A^{\eta}\phi = \frac{\sin(\pi\eta)}{\pi} \int_0^{\infty} s^{\eta-1} A(s + A)^{-1} \phi ds. \tag{6}$$

There are extensions of the above definition valid for the powers $\eta \geq 1$ (e.g. [M-S]), and for negative powers (e.g. [HE]).

The Cauchy integral formula.

An abstract semilinear Cauchy's problem with sectorial positive operator has a solution (called 'mild solution') given by the *Cauchy formula*:

$$u(t,u_0) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}F(u(s,u_0))ds, \quad t \in [0,\tau_{u_0}), \quad (7)$$

τ_{u_0} being the *life time* of the local solution corresponding to the initial data u_0. Here e^{-At} is the semigroup corresponding to the linear equation:

$$\begin{aligned} &u_t + Au = 0, \quad t > 0, \\ &u(0) = u_0. \end{aligned} \quad (8)$$

The integral is understood as a *Bochner integral* with values in a Banach space. Two estimates valid for sectorial positive operators are decisive when applying the Banach theorem to integral equation; for certain positive constants c_0, c_1 and a as in (5)

$$\|e^{-At}\|_{\mathcal{L}(X,X)} \le c_0 e^{-at}, t \ge 0; \quad \|Ae^{-At}\|_{\mathcal{L}(X,X)} \le \frac{c_1}{t} e^{-at}, t > 0. \quad (9)$$

Local in time X^α solutions.

The following local existence result is valid for the mild solutions to

$$\begin{aligned} &u_t + Au = F(u), \quad t > 0, \\ &u(0) = u_0. \end{aligned} \tag{10}$$

The solutions are varying continuously in the phase space X^α:

Theorem

(Dan Henry) Let X be a Banach space, $A : D(A) \to X$ a sectorial positive operator in X and $F : X^\alpha := D(A^\alpha) \to X$, be Lipschitz continuous on bounded subsets of X^α for certain $\alpha \in (0, 1)$. Then, for each $u_0 \in X^\alpha$, there exists a unique mild solution of (3) *defined on its maximal* interval of existence $[0, \tau_{u_0})$, *having the following properties:*

$$u \in C^0([0, \tau_{u_0}), X^\alpha) \cap C^1((0, \tau_{u_0}), X), u(t) \in D(A) \ t \in (0, \tau_{u_0}),$$

$$u_t \in C^0((0, \tau_{u_0}), X^\gamma), \gamma < \alpha.$$

Moreover, the equation (3) *is satisfied in X for all $t \in (0, \tau_{u_0})$.*

In order to study higher regularity of solutions the above local existence result will be naturally generalized. The idea of [Am, Ch-D, Cz-D] was to consider the connected with sectorial positive operator A *fractional powers* $A^{\alpha}, \alpha \in \mathbb{R}$, together with their domains $D(A^{\alpha}) =: X^{\alpha}$.
We are using further the *fractional power scale* of Banach spaces $\{X^{\alpha}\}_{\alpha \in J}$, J being the interval $[0, \infty)$ or $[-1, \infty)$ (see [Am]).
Denoting by A the realization of the positive sectorial operator in a chosen *base space* X^{β}, $\beta \in J$, from the scale, we are considering a semilinear abstract Cauchy problem in X^{β}

$$u_t + Au = F(u), \; t > 0, \quad u(0) = u_0, \tag{11}$$

under the assumption that the nonlinearity

$$\begin{aligned} &F \colon X^{\alpha} \to X^{\beta} \text{ with some } \alpha \in J, \; 0 \le \alpha - \beta < 1, \\ &\text{is Lipschitz continuous on bounded subsets of } X^{\alpha}. \end{aligned} \tag{12}$$

Note that A takes isometrically $X^{\beta+1}$ onto X^{β} and the *phase space* X^{α} is intermediate between $X^{\beta+1}$ and X^{β} with dense and continuous embeddings. Local existence result extends naturally.

Global in time extendibility of local solutions.

Following [Ch-D, Theorem 3.1.1] and [WA], we recall:

Theorem. Let Y be a normed space such that $X^{\beta+1} \subset Y$, u_0 an initial data from the phase space X^α and u the corresponding solution of (11) defined on the maximal interval of existence $[0, \tau_{u_0})$. Let $0 < T \leq \infty$ and $\zeta \in [\alpha, \beta + 1)$. Assume that for each $v_0 \in X^\zeta$ the corresponding solution v of (11) satisfies an a priori estimate;

$$\exists_{\mathcal{C}=\mathcal{C}(v_0,T)>0} \; \|v(t)\|_Y \leq \mathcal{C}, \;\; t \in (0, \min\{\tau_{\zeta,v_0}, T\}) \qquad (13)$$

where τ_{ζ,v_0} denotes the life time in the phase space X^ζ. Let for some $\theta \in [0, 1)$ and a nondecreasing function $g\colon [0, \infty) \to [0, \infty)$

$$\|F(v(t))\|_{X^\beta} \leq g(\|v(t)\|_Y) \left(1 + \|v(t)\|^\theta_{X^\zeta}\right), \; t \in (0, \min\{\tau_{\zeta,v_0}, T\}). \qquad (14)$$

Then u is a bounded solution in X^α defined on $[0, T)$.

Condition (14) express the fact that *the nonlinearity F (on the solution) is controlled through the θ-root of A^α* together with (13).

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Examples of quasilinear problems.

The announced above abstract approach to quasilinear problems was illustrated in [Cz-D] by the following three examples.

1. The parabolic Kirchhoff equation,
$$u_t = M(\int_\Omega |\nabla u|^2 dx)\Delta u + f(u),\ x \in \Omega,\ t > 0, \qquad (15)$$
where the main operator depends on the norm of the solution. We regularize (15) with higher order operator $\epsilon(-\Delta)^{1+\delta}$.
2. A quasilinear reaction-diffusion equation
$$u_t = a(u)\Delta u + f(x, u, \nabla u) \text{ in } (0, T) \times \Omega, \qquad (16)$$
considered under suitable assumptions on a and f.
3. Dirichlet problem for the critical 2D surface quasi-geostrophic equation
$$\theta_t + u \cdot \nabla\theta + \kappa(-\Delta)^{\frac{1}{2}}\theta = f,\ x \in \Omega \subset \mathbb{R}^2,\ t > 0. \qquad (17)$$
By perturbing (17) with $-\epsilon\Delta\theta$, we prove existence of a weak solution defined on an arbitrarily long time interval.

3D Navier-Stokes equations.

While evidently the most exciting application of that technique is dedicated to the 3D Navier-Stokes equation. Dirichlet problem for classical 3-D *Navier-Stokes equations* has the form:

$$
\begin{aligned}
& u_t = \nu \Delta u - \nabla p - (u \cdot \nabla) u + f, \quad div u = 0, \quad x \in \Omega, \ t > 0, \\
& u = 0, \quad t > 0, \quad x \in \partial\Omega, \\
& u(0, x) = u_0(x),
\end{aligned}
\tag{18}
$$

$\nu > 0$ is the viscosity coefficient, $u = (u_1(t, x), u_2(t, x), u_3(t, x))$ denotes velocity, $p = p(t, x)$ pressure, $f = (f_1(x), f_2(x), f_3(x))$ external force, and $\Omega \subset \mathbb{R}^3$ is a bounded domain with C^2 boundary. Applying the projector P to subspace of divergence free functions, problem (18) will be written equivalently as

$$
\begin{aligned}
& u_t + Au = -P(u \cdot \nabla) u + Pf, \quad t > 0, \\
& u(0) = u_0.
\end{aligned}
\tag{19}
$$

Proper choice of the level (at the fractional power scale associated with the Stokes operator) at which the N-S equation will be settled is important for simplicity of construction of the solution. Vital here are estimates of nonlinear term, due to [G-M, Lemma 2.1], which in a simplest case reads:

Corollary

For each $j, 1 \leq j \leq N$, the operator $A^{-\frac{1}{2}} P \frac{\partial}{\partial x_j}$ extends uniquely to a bounded linear operator from $[L^r(\Omega)]^N$ to $X_r, 1 < r < \infty$. Consequently, the following estimate holds:

$$\|A^{-\frac{1}{2}} P(u \cdot \nabla) v\|_{[L^r(\Omega)]^N} \leq M(r) \||u||v|\|_{[L^r(\Omega)]^N}. \tag{20}$$

We formulate next an important estimate needed further in the text, a consequence of Corollary 3. For all $N \in \mathbb{N}$:

$$\|A^{-\frac{1}{2}} P(u \cdot \nabla) v\|_{[L^2(\Omega)]^N} \leq c\||u||v|\|_{[L^2(\Omega)]^N} \leq c\|u\|_{[L^4(\Omega)]^N} \|v\|_{[L^4(\Omega)]^N},$$
$$\|P(u \cdot \nabla) v\|_{[L^2(\Omega)]^N} \leq c\|u\|_{[L^4(\Omega)]^N} \|\nabla v\|_{[L^4(\Omega)]^N}. \tag{21}$$

A generalization of the classical Navier-Stokes equations containing higher order viscosity term was proposed in 1969 by J.-L. Lions; [Li, Chapt.1]. We consider here the regularization:

$$u_t = -Au - \epsilon A^s u + F(u) + Pf, \tag{22}$$

with $s > \frac{5}{4}$ (close to $\frac{5}{4}$), and denote its solution by u^ϵ. Applying to (22) operator $A^{-s+\frac{1}{2}}$ we get

$$A^{-s+\frac{1}{2}} u_t^\epsilon = -A^{-s+\frac{3}{2}} u^\epsilon - \epsilon A^{\frac{1}{2}} u^\epsilon + A^{-s+\frac{1}{2}} F(u^\epsilon) + A^{-s+\frac{1}{2}} Pf. \tag{23}$$

Using to nonlinearity the estimate of [G-M, Lemma 2.2] with $\delta = s - \frac{1}{2}, \epsilon = s - 1$, we get

$$\begin{aligned} &\|A^{-\delta} F(u^\epsilon)\|_{[L^2(\Omega)]^3} \le c \| |u^\epsilon|^2 \|_{[L^z(\Omega)]^3} \le c' \|u^\epsilon\|^2_{[L^{2z}(\Omega)]^3} \\ &\le c'' \|u^\epsilon\|^{\frac{7}{2}-2s}_{[H^1(\Omega)]^3} \|u^\epsilon\|^{2s-\frac{3}{2}}_{[L^2(\Omega)]^3} = c''' \|A^{\frac{1}{2}} u^\epsilon\|^{\frac{7}{2}-2s}_{[L^2(\Omega)]^3} \|u^\epsilon\|^{2s-\frac{3}{2}}_{[L^2(\Omega)]^3}, \end{aligned} \tag{24}$$

where $z = \frac{6}{3+4(s-1)} < \frac{3}{2}$ (but close). By the standard $[L^2(\Omega)]^3$ estimate, all the right hand side components in (23) belong to $L^2(0, T; [L^2(\Omega)]^3)$; note that $\frac{7}{2} - 2s < 1$ in (24)). Consequently,

$$A^{-s+\frac{1}{2}} u_t^\epsilon \in L^2(0, T; [L^2(\Omega)]^3), \text{ or } u_t^\epsilon \in L^2(0, T; D(A^{-s+\frac{1}{2}})).$$

Using weak formulation of the problems (22) and the mentioned uniform in $\epsilon > 0, \delta > 0$ a priori estimates we are able to let $\epsilon \to 0$ and obtain a weak solution to the original Navier-Stokes equations in such limit; moreover, this is the classical J. Leray's weak solution from [LE]. Since the procedure of passing to the limit is standard nowadays, we will skip it here showing however what will hapened with the extra term (added to the equation); it follows from the standard L^2 a priori estimate for (22) that:

$$\sqrt{\epsilon}\|A^{\frac{s}{2}}u^\epsilon\|_{L^2(0,T;[L^2(\Omega)]^3)} \leq const, \tag{25}$$

with const independent on $\epsilon > 0$. We recall next a *weak formulation* of the approximating equations (22):

$$< u_t^\epsilon, v >_{[L^2(\Omega)]^3} = - < A^{\frac{1}{2}}u^\epsilon, A^{\frac{1}{2}}v >_{[L^2(\Omega)]^3} - \epsilon < A^{\frac{s}{2}}u^\epsilon, A^{\frac{s}{2}}v >_{[L^2(\Omega)]^3}$$
$$+ < F(u^\epsilon), v >_{[L^2(\Omega)]^3} + < Pf, v >_{[L^2(\Omega)]^3},$$

with arbitrary 'test function' $v \in D(A^{\frac{s}{2}})$. Uniform with respect to parameters $\epsilon > 0, s > \frac{5}{4}$ a priori estimates allow to let $\epsilon \to 0$ in the last formula, obtaining a weak solution of the original N-S equation in the limit. We omit technical details here.

Kirchhoff equation.

Due to time limitation I will draw only one more application of that technique to the Kirchhoff equation (15). The problem proposed to approximate solutions of quasilinear equation (15) reads:

$$
\begin{aligned}
&u_t^{\epsilon} = -\epsilon(-\Delta)^{1+\delta}u^{\epsilon} + M\Big(\int_{\Omega} |\nabla u^{\epsilon}|^2 dx\Big)\Delta u^{\epsilon} + f(u^{\epsilon}), \\
&u^{\epsilon}(t,x) = 0,\ x \in \partial\Omega,\ t > 0, \quad u^{\epsilon}(0,x) = u_0(x),\ x \in \Omega,
\end{aligned}
\tag{26}
$$

where, to avoid introducing extra boundary conditions we assume that $0 < \delta < \frac{1}{4}$ is fixed and also that $\epsilon \in (0,1)$.
Proposed approximation is a semilinear parabolic problem, and we are able to apply the standard unified semigroup approach (see [HE, Ch-D]). Note that the operator $M(\int_{\Omega} |\nabla u|^2 dx)\Delta u$ will be treated here as a part of the nonlinearity. There are also good a priori estimates both for the original problem (15) and for (26), allowing to let $\epsilon \to 0$ and get a weak solution of (15) in the limit.

Assuming that; Ω is a bounded C^2 domain in $\mathbb{R}^N$, $N \geq 1$, $M\colon [0, +\infty) \to \mathbb{R}$ is a locally Lipschitz continuous satisfying

$$M(s) \geq m_0 > 0 \text{ for any } s \geq 0, \tag{27}$$

$f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz, and

$$|f(s_1) - f(s_2)| \leq c_0 |s_1 - s_2| (1 + |s_1|^{\rho-1} + |s_2|^{\rho-1}), \; s_1, s_2 \in \mathbb{R}, \tag{28}$$

holds with $1 \leq \rho \leq \frac{N}{N-2}$ for $N \geq 3$ and arbitrary $\rho \geq 1$ for $N = 2$. Consequently, the nonlinearity $F(u) = M(\|\nabla u\|^2_{L^2(\Omega)})\Delta u + f(u)$ is a Lipschitz map on bounded subsets of $X^{\frac{1}{1+\delta}} = D(-\Delta) \subset H^2(\Omega)$ with values in $X^0 = L^2(\Omega)$.

Theorem

Assume that (27) and (28) hold. For arbitrarily fixed $\delta \in (0, \frac{1}{4})$ and any $\epsilon \in (0, 1)$ and $u_0 \in D(-\Delta)$ there exists a unique local in time solution u^ϵ to the approximating problem (26) enjoying the following regularity properties:

$$u^\epsilon \in C([0, \tau_{u_0}); D(-\Delta)) \cap C((0, \tau_{u_0}); D((-\Delta)^{1+\delta})),$$

$$u^\epsilon_t \in C((0, \tau_{u_0}); D((-\Delta)^{(1+\delta)^-})),$$

Standard procedure gives a priori estimate in $H_0^1(\Omega)$. Multiplying (26) by u_t^ϵ and integrating we find a *Lyapunov function*:

$$\int_\Omega (u_t^\epsilon)^2 dx = -\frac{d}{dt}\Big(\frac{\epsilon}{2}\int_\Omega [(-\Delta)^{\frac{1+\delta}{2}} u^\epsilon]^2 dx + \frac{1}{2}\mathcal{M}(\|\nabla u^\epsilon\|^2_{L^2(\Omega)}) - \int_\Omega \mathcal{F}(u^\epsilon) dx\Big),$$

where $\mathcal{F}(s) = \int_0^s f(\sigma)d\sigma$ is a primitive of f and $\mathcal{M}(s) = \int_0^s M(\sigma)d\sigma$ a primitive of M. Consequently, for $v \in D((-\Delta)^{\frac{1+\delta}{2}})$, the function

$$\mathcal{L}_\epsilon(v) := \frac{\epsilon}{2}\int_\Omega [(-\Delta)^{\frac{1+\delta}{2}} v]^2 dx + \frac{1}{2}\mathcal{M}(\|\nabla v\|^2_{L^2(\Omega)}) - \int_\Omega \mathcal{F}(v) dx,$$

is nonincreasing in time along the solution $u^\epsilon(t,x)$, that is,

$$\mathcal{L}_\epsilon(u^\epsilon(t,\cdot)) \le \mathcal{L}_\epsilon(u_0),\ \ t \in [0, \tau_{u_0}). \tag{29}$$

By (27) $\mathcal{M}$, the primitive of M, satisfies

$$\mathcal{M}(s) = \int_0^s M(\sigma)d\sigma \geq m_0 s, \;\; s \in \mathbb{R}. \tag{30}$$

To control the $H_0^1(\Omega)$ norm, we require that $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} \leq 0. \tag{31}$$

Then, as shown e.g. in [Ha, p.76], for any $\nu > 0$ there exists $C_\nu > 0$ such that

$$\mathcal{F}(s) = \int_0^s f(\sigma)d\sigma \leq \nu s^2 + C_\nu, \;\; s \in \mathbb{R}. \tag{32}$$

With the above assumptions a uniform in $\epsilon \in (0,1)$, valid for any $u_0 \in D(-\Delta)$ a priori estimate holds:

$$\|u^\epsilon\|_{C([0,\infty);H_0^1(\Omega))} + \|\sqrt{\epsilon}u^\epsilon\|_{C([0,\infty);D((-\Delta)^{\frac{1+\delta}{2}}))} \leq \mathcal{C}, \;\; \epsilon \in (0,1), \tag{33}$$

with a positive constant $\mathcal{C}$ independent of ϵ.

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THANK YOU FOR YOUR ATTENTION