Triple junction solution for the Allen-Cahn system

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Energy functional and Hypothesis

Consider the energy functional

$$J(u,\Omega) := \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + W(u)\right) dx, \quad \forall \Omega \subset \mathbb{R}^2, \; u : \Omega \to \mathbb{R}^2. \tag{1}$$

The Euler-Lagrange equation is

$$\Delta u - W_u(u) = 0, \tag{2}$$

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<u>Hypotheses on W:</u>

(H1). $W \in C^2(\mathbb{R}^2; [0,+\infty))$, $\{z : \ W(z) = 0\} = \{a_1, a_2, a_3\}$, and

$$c_2|\xi|^2 \geq \xi^T W_{uu}(a_i)\xi \geq c_1|\xi|^2, \ i = 1,2,3.$$

(H2). <u>Existence of heteroclinic connections:</u> $\forall i \neq j$, $\exists U_{ij} \in W^{1,2}(\mathbb{R}, \mathbb{R}^2)$ be an 1D minimizer of

$$\sigma_{ij} := \min \int_{\mathbb{R}} \left(\frac{1}{2}|U'|^2 + W(U)\right) d\eta, \quad \lim_{\eta\to-\infty} U(\eta) = a_i, \ \lim_{\eta\to+\infty} U(\eta) = a_j.$$

σ_{ij} satisfies

$$\sigma_{ij} < \sigma_{ik} + \sigma_{jk}, \quad \forall \{i,j,k\} = \{1,2,3\}. \tag{3}$$

DIRICHLET PROBLEM ON THE UNIT DISK

$$\min_{u\in\mathcal{A}}\int_{B_1(0)}\left(\frac{\varepsilon}{2}|\nabla u|^2+\frac{1}{\varepsilon}W(u)\right)\,dx=:\min_{u\in\mathcal{A}}J_\varepsilon(u), \tag{4}$$

within the admissible set

$$\mathcal{A}=\{u\in W^{1,2}(B_1):u=g_\varepsilon\text{ on }\partial B_1\}.$$

g_ε is a smooth function connecting the three phases in $O(\varepsilon)$ intervals.

$$I_{\varepsilon,\gamma}:=\{x\in B_1:|u_\varepsilon(x)-a_i|>\gamma,\ \forall i\}.$$

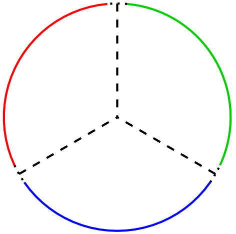


Figure. g_ε on ∂B_1.

Problem: geometric/analytic description of the diffuse interface $I_{\varepsilon,\gamma}$.

Entire solution

Blowup: $u_\varepsilon(\varepsilon x + x_0)$. Pick the blowup center x_0 as the approximate location of the junction point. As $\varepsilon \to 0$, one expects that it converges to an entire minimizing solution of

$$\Delta u - W_u(u) = 0, \quad u : \mathbb{R}^2 \to \mathbb{R}^2$$

connecting the three phases at infinity.

Problem: asymptotic behavior of u at infinity.

$$u(rx) \xrightarrow[r\to\infty]{L^1_{loc}(\mathbb{R}^2)} u_{\mathcal{P}}(x) = \sum_{i=1}^{3} a_i \chi_{D_i}.$$

Minimizing partition of $\mathbb{R}^2$

Let

$$\mathcal{P} = \{D_1, D_2, D_3\},$$

which is a partition of the plane into three sectors with angles $\alpha_1,\ \alpha_2,\ \alpha_3$, which satisfies Young's law, that is

$$\frac{\sin\alpha_1}{\sigma_{23}} = \frac{\sin\alpha_2}{\sigma_{13}} = \frac{\sin\alpha_3}{\sigma_{12}}. \tag{5}$$

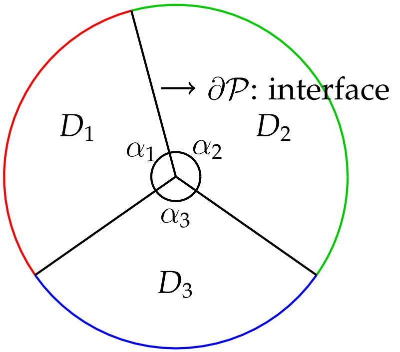


Figure. The partition $\mathcal{P} = \{D_1, D_2, D_3\}$

$\mathcal{P}$ is a minimizing partition of $\mathbb{R}^2$ that minimizes the following functional:

$$\min \sum_{i<j} \sigma_{ij}\mathcal{H}^1(\partial(D_i \cap \Omega) \cap \partial(D_j \cap \Omega)).$$

When $\sigma_{ij} \equiv \sigma$, $\alpha_i = 120^\circ$ for $i = 1, 2, 3$. Triple junction map: $u_{\mathcal{P}} = \sum_{i=1}^{3} a_i \chi_{D_i}$.

Related result: Gamma convergence

(Fonseca-Tartar, 89′), (Baldo, 90′), (Sternberg-Zeimer, 94′), (Gazoulis, 22′): Let u_ε be a minimizer of the Dirichlet problem,

$$\lim_{\varepsilon \to 0} \|u_\varepsilon - u_{\mathcal{P}}\|_{L^1(B_1)} = 0,$$

where $u_{\mathcal{P}} = \sum_{i=1}^{3} a_i \chi_{D_i}$.

<u>Limitations</u>:

- No rate of convergence for

$$\lim_{\varepsilon \to 0} \int_{B_1} \left(\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \right) dx = \sum_{i \neq j} \sigma_{ij}.$$

- Does not yield a nontrivial blow-up limit near the triple junction.
- No monotonicity formula for the Allen-Cahn system (Contrary to the scalar case).

Related result: convergence of u_ε

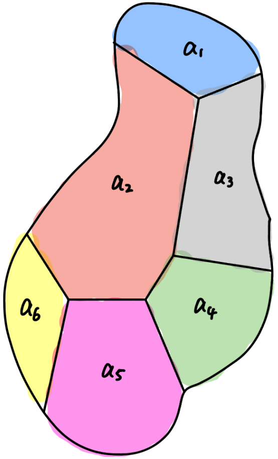
For 2D domain Ω, for the general setting with several energy wells $\{a_1, ..., a_k\}$, let u_ε be a solution (not minimizing) of

$$\Delta u_\varepsilon = \frac{1}{\varepsilon^2} W_u(u_\varepsilon), \quad u : \Omega \to \mathbb{R}^2.$$

Theorem (Bethuel, 20')

There exists a 1D rectifiable subset Γ and a singular set $\mathcal{S} \subset \Gamma$ such that

1. $u_{\varepsilon_n} \to a_i$ *uniformly on every connected compact set of* $\Omega \setminus \Gamma$.
2. $\mathcal{H}^1(\mathcal{S}) = 0$; *for every* $x_0 \in \Gamma \setminus \mathcal{S}$, Γ *is locally a segment.*
3. $\frac{W(u_{\varepsilon_n})}{\varepsilon_n} \to \zeta$ *in the sense of measures on* Ω, *for some measure* ζ *concentrating on* Γ.
4. *Define the 1D density* $\Theta(x) = \liminf_{r\to 0} \frac{\zeta(B(x,r))}{2r}$. *The rectifiable one-varifold* $V(\Gamma, \Theta)$ *corresponding to the measure* ζ *is stationary.*

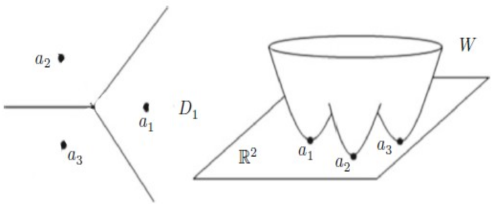


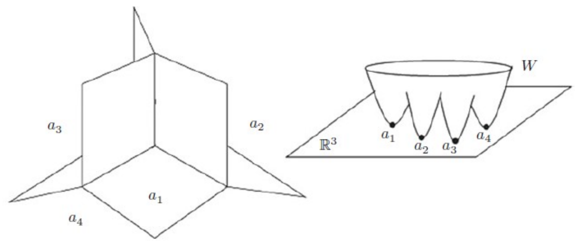
Related result: symmetric solutions

Junctions.

- $W \circ g = W$, $u \circ g = g \circ u$, $g \in G$ that is the group of isometries of the equilateral triangle, (Bronsard-Gui-Schatzman, 96′).
- Triple junction in equivalent class of rotation group, (Fusco, 22′).
- $n = 3$, W is a 4-well potential, existence of solution that is invariant with respect to the symmetries of the tetrahedron, (Gui-Schatzman, 08′).

These solutions are *not stable* under general perturbations.





Energy lower & upper bounds for Dirichlet problem

Proposition 1 (Alikakos-G, 24')

There exists a constant $C(W)$ such that for $\varepsilon \ll 1$,

$$\sigma_{12}+\sigma_{23}+\sigma_{13}-C\varepsilon^{\frac{1}{2}} \leq \int_{B_1}\left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2+\frac{1}{\varepsilon}W(u_\varepsilon)\right)dx \leq \sigma_{12}+\sigma_{23}+\sigma_{13}+C\varepsilon. \tag{6}$$

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- Upper bound: direct construction of an energy competitor.
- Lower bound: estimate the energy by integrating the energy over 1D slices from different directions.

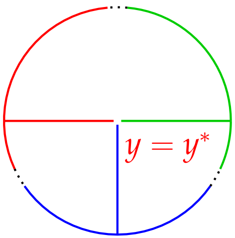

Figure. red: a_1; green: a_2; blue: a_3

Suppose $\exists\, y^*$ such that on $\{y = y^*\}$, u_ε equals to a_1 on the left half and a_2 on the right half. And in the region $\{y < y_*\}$ there is a vertical line on which u_ε equals to a_3.
Then on each subdomain, the boundary condition implies the lower bound $\sigma_{12} + \sigma_{23} + \sigma_{13}$.
The key is to "find" these interior boundaries.

Localization of $I_{\varepsilon,\gamma}$

We use the upper/lower bounds to show that the diffuse interface $I_{\varepsilon,\gamma}$ is contained in an $O(\varepsilon^{\frac{1}{4}})$ neighborhood of the sharp interface $\partial\mathcal{P}$.

Proposition 2 (Alikakos-G, 2024)

$\exists$ a constant γ_0 s.t. for any $0 < \gamma \leq \gamma_0$, $\exists$ constants $C = C(\gamma, W)$

$$I_{\varepsilon,\gamma} \subset N_{C\varepsilon^{\frac{1}{4}}}(\partial\mathcal{P}), \quad \forall \varepsilon \ll 1.$$

Moreover, $\exists$ positive K and k s.t.

$$|u(x) - a_i| \leq Ke^{-\frac{k}{\varepsilon}(\text{dist}(x,\partial\mathcal{P})-C\varepsilon^{\frac{1}{4}})^+}, \quad x \in D_i, \ \forall i.$$

Idea of the proof:

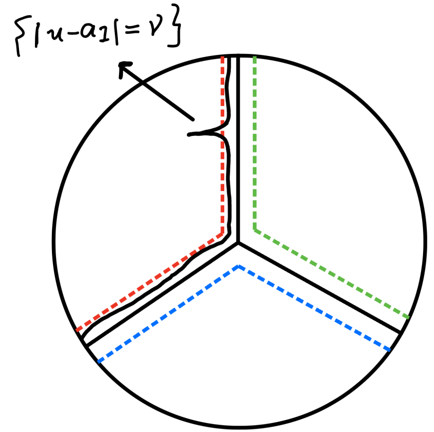

Figure. The deviation of the level set $\{|u - a_1| = \gamma\}$ from the limit interface $\partial\mathcal{P}$ generates large energy in 2D.

Width of the transition layer

Set

$$\Gamma^i_{\varepsilon,\gamma} := \{x \in \bar{B}_1 : \; |u_\varepsilon(x) - a_i| = \gamma\}.$$

$$\partial I_{\varepsilon,\gamma} \subset \bigcup_{i=1}^{3} \Gamma^i_{\varepsilon,\gamma}.$$

Proposition 3 ($O(\varepsilon)$ width of the transition layer)

Fix small γ. $\exists\, C = C(\gamma, W)$ s.t. for any $i \in \{1,2,3\}$ and $\varepsilon \ll 1$,

$$\Gamma^i_{\varepsilon,\gamma} \subset N_{C\varepsilon}\Big(\bigcup_{j\neq i} \Gamma^j_{\varepsilon,\gamma}\Big).$$

- Tool: vector version of Caffarelli-Córdoba density estimate.

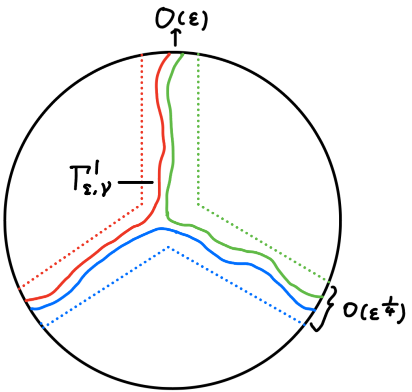

Figure. The width of $I_{\varepsilon,\gamma}$ is $O(\varepsilon)$.

Theorem (Alikakos-G, 24′)

There is an entire, bounded minimizing solution such that, along a sequence $r_k \to \infty$,

$$u(r_k x) \to u_{\mathcal{P}}(x) \ \ in\ L^1_{loc}(\mathbb{R}^2), \tag{7}$$

where $u_{\mathcal{P}} = \sum_{i=1}^{3} a_i \chi_{D_i}$. $\mathcal{P} = \{D_1, D_2, D_3\}$ *gives a minimal partition of* $\mathbb{R}^2$.

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- $\exists\, P_\varepsilon,\ Q_\varepsilon,\ R_\varepsilon$ s.t.

$$u(P_\varepsilon) \sim a_1,\ u(Q_\varepsilon) \sim a_2,\ u(R_\varepsilon) \sim a_3,$$
$$\mathrm{dist}(P_\varepsilon, Q_\varepsilon), \mathrm{dist}(Q_\varepsilon, R_\varepsilon) \sim O(\varepsilon).$$

- For every r, $\exists\, P(r)$ s.t. $\mathrm{dist}(P(r), P_\varepsilon) = r$, $u(P(r)) \sim a_1$.
- $\exists \{Q_j\},\ \{R_j\}$ s.t. $u(Q_j) \sim a_2,\ u(R_j) \sim a_3$, and $\mathrm{dist}(Q_j, P_\varepsilon) \leq Cj\varepsilon,\ \mathrm{dist}(R_j, P_\varepsilon) \leq Cj\varepsilon$.

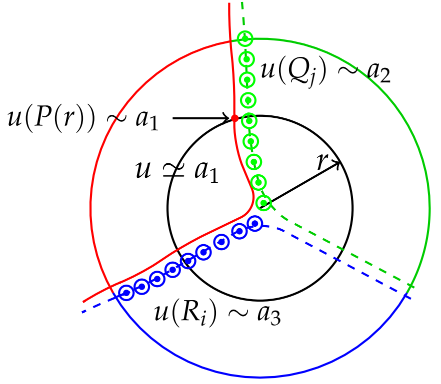

Figure. Triplets of points, close to a_1, a_2, a_3 respectively.

Roughly at the same time, Sandier and Sternberg obtained comparable results. Under the assumption that the heteroclinic connection U_{ij} is unique, they proved:

Theorem (Sandier-Sternberg, 24′)

For any $r_k \to \infty$ (up to a possible subsequence),

$$u(r_k x) \to u_{\mathcal{P}} \text{ in } L^1_{loc}(\mathbb{R}^2),$$

where $u_{\mathcal{P}}$ is a triple junction map which might depend on $\{r_k\}$.

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- Pohozaev identity.

$$\frac{d}{dR}\left(\frac{1}{R}\int_{B_R} W(u)\right) = \frac{1}{2R}\int_{\partial B_R}\left(\frac{1}{2}|u_\nu|^2 - \frac{1}{2}|u_s|^2 + W(u)\right)$$

- Asymptotic energy equipartition.

$$\left|\int_{B_R}(\sqrt{W(u)} - \frac{1}{\sqrt{2}}|\nabla u|)^2\right| \leq CR^\alpha, \quad \alpha \in (0,1).$$

Uniqueness of the blow-down limit

Theorem (G, 24')

There exists a minimizing partition $\mathcal{P} = \{D_i\}_{i=1}^3$ *such that*

$$\lim_{r\to\infty} \|u(rx) - u_{\mathcal{P}}\|_{L^1(B_1)} = 0,$$

where $u_{\mathcal{P}} = \sum_{i=1}^{3} a_i \chi_{D_i}$ *is the unique blow-down limit at infinity.*

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Corollary (Sharp energy bounds)

There exists a constant $C = C(u)$ *such that for any* $R > 0$*,*

$$R(\sigma_{12} + \sigma_{23} + \sigma_{13}) - C \leq \int_{B_R} \left(\frac{1}{2}|\nabla u|^2 + W(u) \right) dx \leq R(\sigma_{12} + \sigma_{23} + \sigma_{13}) + C.$$

Sketch of the proof

1. For any R sufficiently large, $\exists\, R_0 \in [R, 2R]$ such that u satisfies a well-behaved boundary condition on ∂B_{R_0}, i.e. $u|_{\partial B_{R_0}}$ is close to a triple junction map.

2. Rescale B_{R_0} to B_1. Utilize the energy lower & upper bounds to localize the diffuse interface of u_{R_0} in a $R_0^{-\alpha}$ neighborhood of the optimal triple junction map $\bar{u}_{R_0}$.

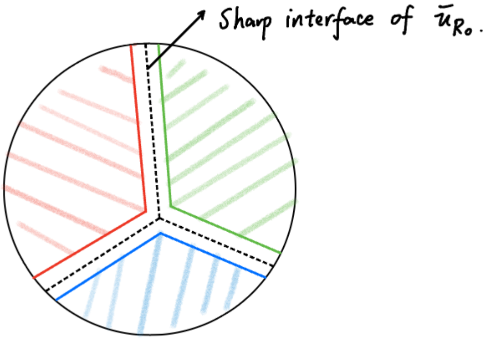

Figure. Optimal triple junction map $\bar{u}_{R_0}$ on B_{R_0}

3. Compare the optimal triple junction map $\bar{u}_{R_1}$ for $R_1 \in [2R, 4R]$ with $\bar{u}_{R_0}$ to obtain

$$\|\bar{u}_{R_0} - \bar{u}_{R_1}\|_{L^1(B_1)} \leq CR_0^{-\alpha}.$$

4. Iterate the above estimate to all the scales $R_k \in [2^k R, 2^{k+1} R]$ to get

$$\|\bar{u}_{R_i} - \bar{u}_{R_j}\|_{L^1(B_1)} \leq C2^{-\alpha \cdot \min\{i,j\}},$$

which converges to 0 as i, j tend to ∞.

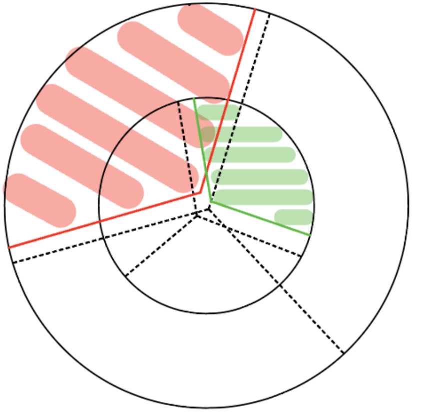

Figure. Optimal triple junctions at two consecutive scales must be close to each other

Asymptotic flatness of the diffuse interface at infinity

By assuming the uniqueness of the heteroclinic connection U_{ij}, we show that u is almost invariant along the direction of the sharp interface.

Theorem (G, 24')

Suppose the 1D heteroclinic connection U_{ij} of two phases is unique. Then for any $i \neq j$, there exists a constant h such that

$$\lim_{x\to+\infty} \|u(x\mathbf{e}_{ij} + y\mathbf{e}_{ij}^{\perp}) - U_{ij}(y-h)\|_{C^{2,\alpha}(\mathbb{R})} = 0,$$

where $\mathbf{e}_{ij}$ is the unit vector parallel to $\partial D_i \cap \partial D_j$.

Sketch of the proof

WLOG, let $\mathbf{e}_{ij} = (1,0)$, $\mathbf{e}_{ij}^{\perp} = (0,1)$.

1. For any $x \gg 1$, $\exists!\, h(x)$ s.t. $\|u(x,y) - U_{ij}(y - h(x))\|_{L^2}$ is minimized (Schatzman 02′).
2. Direct calculation yields

$$|h'(x)| = \left| \frac{\int_{\mathbb{R}} \partial_x u \cdot U_{ij}'(y - h(x))\, dy}{\int_{\mathbb{R}} \Big(|U_{ij}'|^2 + U_{ij}''(y - h(x))(u(x,y) - U_{ij}(y - h(x))) \Big)\, dy} \right|$$
$$\leq C \int_{-\infty}^{\infty} |\partial_x u(x,y)|^2\, dy$$

3. Use the tight energy bounds to show

$$\int_{x>0} |\partial_x u(x,y)|^2\, dxdy < \infty$$

.

A vector version of De Giorgi conjecture

Suppose $u : \mathbb{R}^2 \to \mathbb{R}^2$ is a minimizing solution of $\Delta u = W_u(u)$, where W is a double-well potential. Assume the uniqueness of the heteroclinic connection U_{12}.

Theorem (Sandier-Sternberg, 24′; G, 24′)

If for each $x \in \mathbb{R}$,

$$\lim_{y \to \infty} u(x, y) = a_1, \quad \lim_{y \to -\infty} u(x, y) = a_2,$$

then u is a one-dimensional solution, i.e. there exists a unit vector $\mathbf{e} \neq (1, 0)$ and a constant h such that

$$u(z) = U_{12}(z \cdot \mathbf{e} - h), \ \forall z \in \mathbb{R}^2.$$

FURTHER DIRECTIONS

1. 2D problems:
   - Symmetry of potential implies symmetry of solution?
   - For unequal σ_{ij}, construct N-junction ($N > 3$) solutions.
2. 3D problems:
   - For a triple-well potential, does a minimizing solution necessarily be a cylindrical triple junction?
   - For a quadruple-well potential, the existence of an entire solution with a quadruple junction profile.
3. Gradient flow: motion of the junction point.

Thank You!