# Front Propagation through a Perforated Wall

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# Outline

1. Introduction
2. Formulation of the problem
3. Propagation / blocking dichotomy
4. Sufficient conditions for blocking
5. Sufficient conditions for propagation

# 1. Introduction

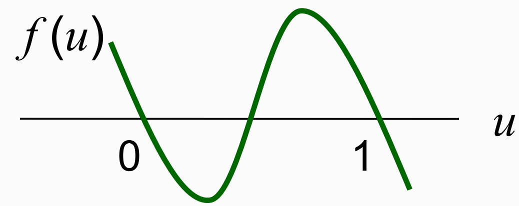
Joint work with

Henri Berestycki (EHESS)

François Hamel (Aix-Marseille)

# Bistable RD equation

$$\frac{\partial u}{\partial t} = \Delta u + f(u) \quad \text{on} \quad \mathbb{R}^N$$




$f(u)$: bistable

$$f(0) = f(\alpha) = f(1) = 0, \quad f'(0) < 0, \; f'(1) < 0$$

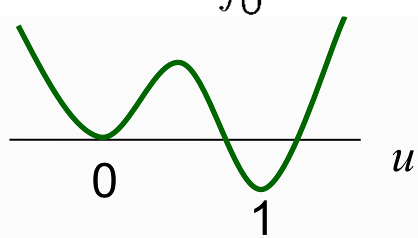
$$\boxed{\int_0^1 f(s)ds > 0}$$

unbalanced

Existence of traveling wave

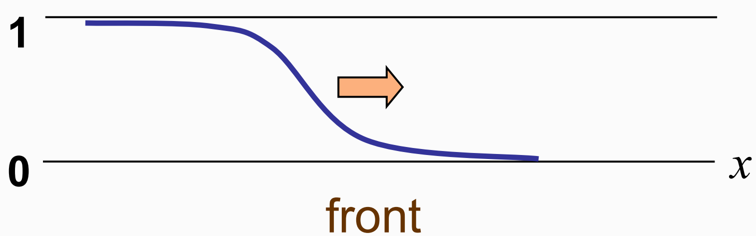
potential

$$W(u) = -\int_0^u f(s)ds$$




unequal well-depth

1D $\quad u_t = u_{xx} + f(u) \quad u(x,t) = \phi(x - ct), \; c > 0$



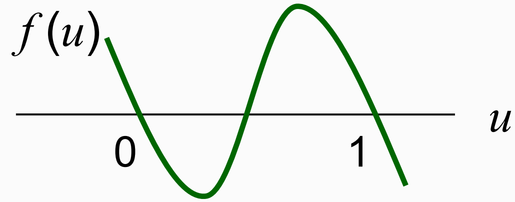

: TW profile

$$\begin{cases} \phi''(z) + c\phi'(z) + f(\phi(z)) = 0 \quad (z \in \mathbb{R}), \\ 0 < \phi < 1, \quad \phi(-\infty) = 1, \; \phi(+\infty) = 0. \end{cases}$$

Fife & McLeod (ARMA1977)

# Bistable RD equation

$$\frac{\partial u}{\partial t} = \Delta u + f(u) \quad \text{on} \quad \mathbb{R}^N$$




$f(u)$: bistable

$$f(0) = f(\alpha) = f(1) = 0, \quad f'(0) < 0, \quad f'(1) < 0$$

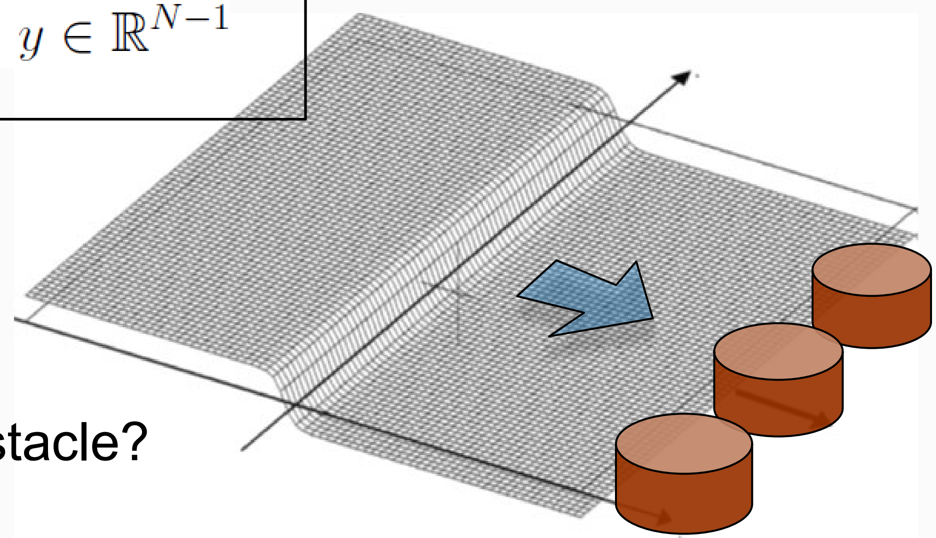
$$\boxed{\int_0^1 f(s)ds > 0}$$

unbalanced

Existence of traveling wave

$\geq 2$ Planar wave $u(x,t) = \phi(x_1 - ct)$

$$x = (x_1, x_2, \ldots, x_N) = (x_1, y), \ \ y \in \mathbb{R}^{N-1}$$



Q What happens if there is an obstacle?

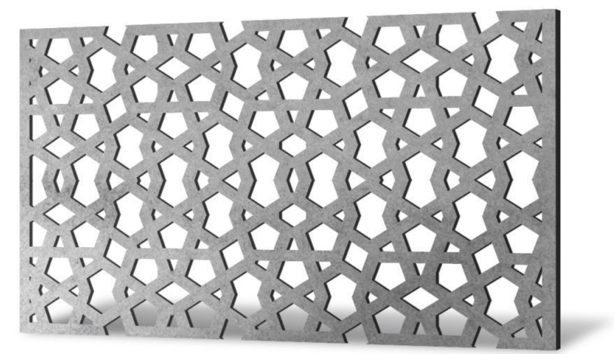
# Description of the obstacle

Obstacle (wall)

$$K \subset \{x \in \mathbb{R}^N \mid 0 \le x_1 \le M\}$$

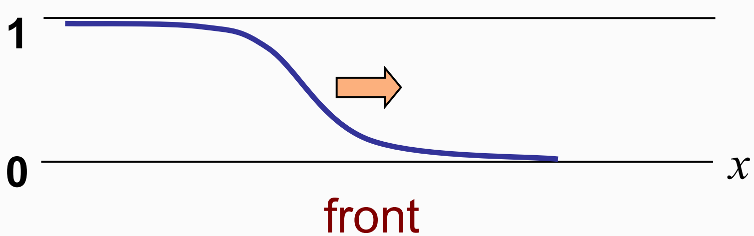
finite thickness

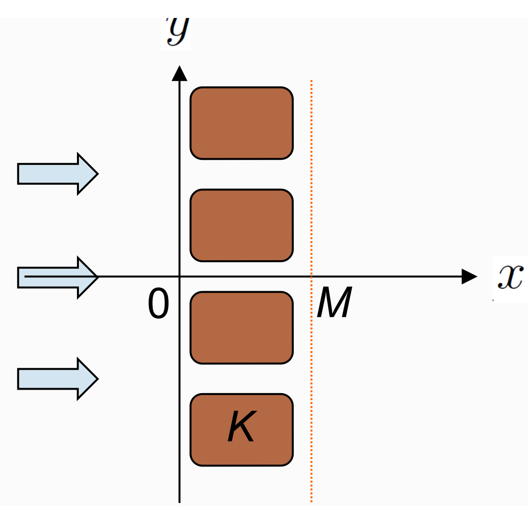
$\Omega := \mathbb{R}^N \setminus K$ connected $\partial\Omega$ smooth



For some results we also assume periodicity

$$\begin{aligned} &{}^{\exists}\boldsymbol{p}_2, \ldots {}^{\exists}\boldsymbol{p}_N \in \mathbb{R}_y^{N-1} \ \text{(linearly independent)} \\ &s.t. \quad K + \boldsymbol{p}_i = K \quad (i = 2, \ldots, N) \end{aligned}$$



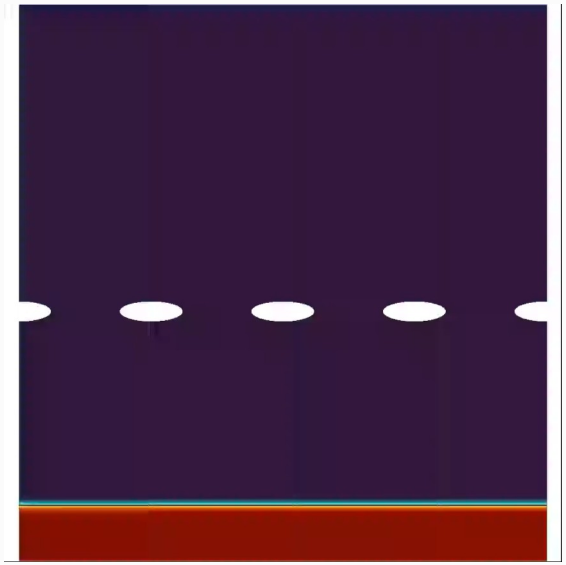




**Q** Under what conditions can the front pass through the wall?

# Some numerical simulations

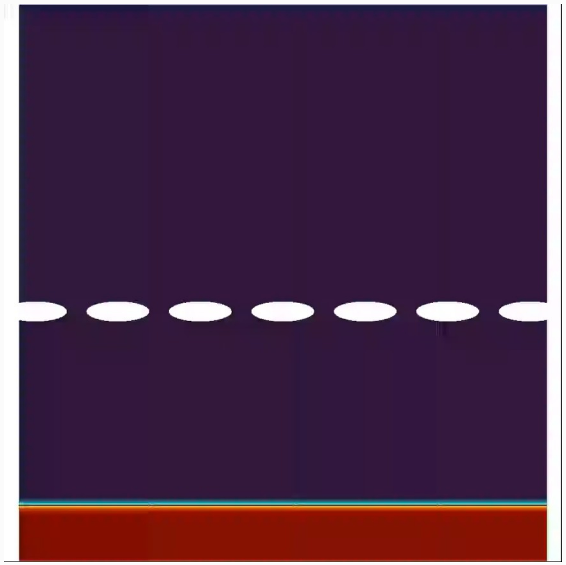
Simulation by Steffen Plunder
(Kyoto University)

Wall with wider holes



Propagation

Wall with narrower holes



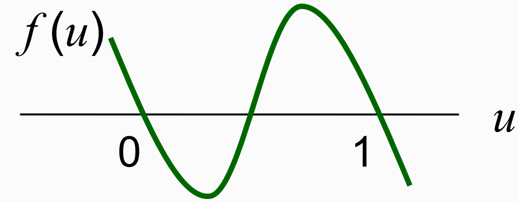
Blocking

The front has surface energy. ⇨ surface tension

# Bistable RD equation & interface motion

$$\frac{\partial u}{\partial t} = \Delta u + f(u)$$

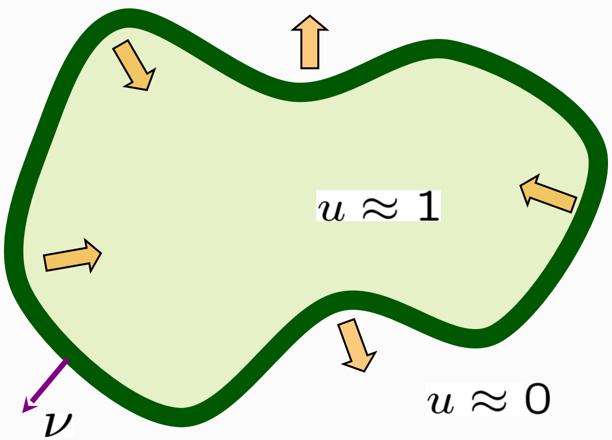
$f(u)$ : bistable




balanced: $\int_0^1 f(s)ds = 0$

unbalanced: $\boxed{\int_0^1 f(s)ds > 0}$ or $\int_0^1 f(s)ds < 0$

## Interface motion ()




outward normal vector

## Approximate law of motion

$f$: unbalanced $V = (N - 1)H + A$

$V$ = normal velocity
$H$ = mean curvature
$N$ = space dimension
$A$ : driving force

Energy $\boxed{E(u) := \int \left( \boxed{\frac{1}{2}|\nabla u|^2} + \boxed{W(u)} \right) dx}$

Interfacial energy
+ bulk energy

*Athens 2025*

# Bistable RD equation & interface motion

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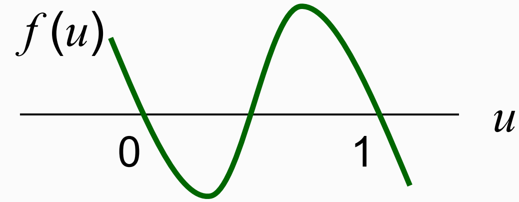
$f(u)$ : bistable

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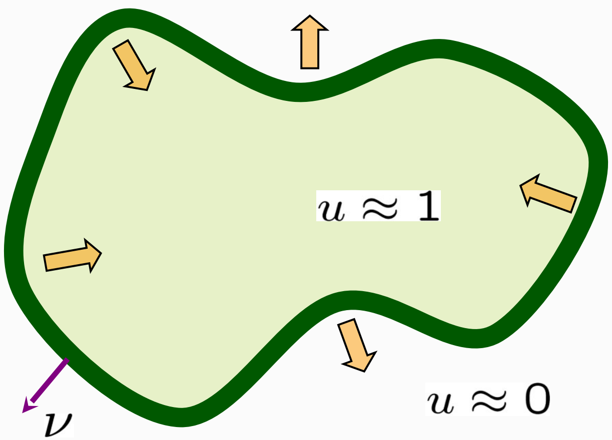
## Interface motion ()

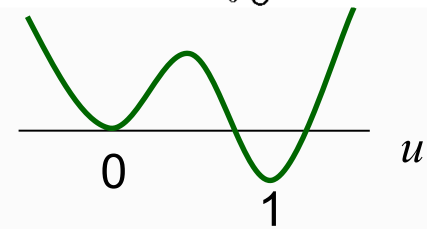



outward normal vector

## Approximate law of motion

$f$: unbalanced $\quad V = (N - 1)H + A$

$$W(u) = -\int_0^u f(s)ds$$




unequal well-depth

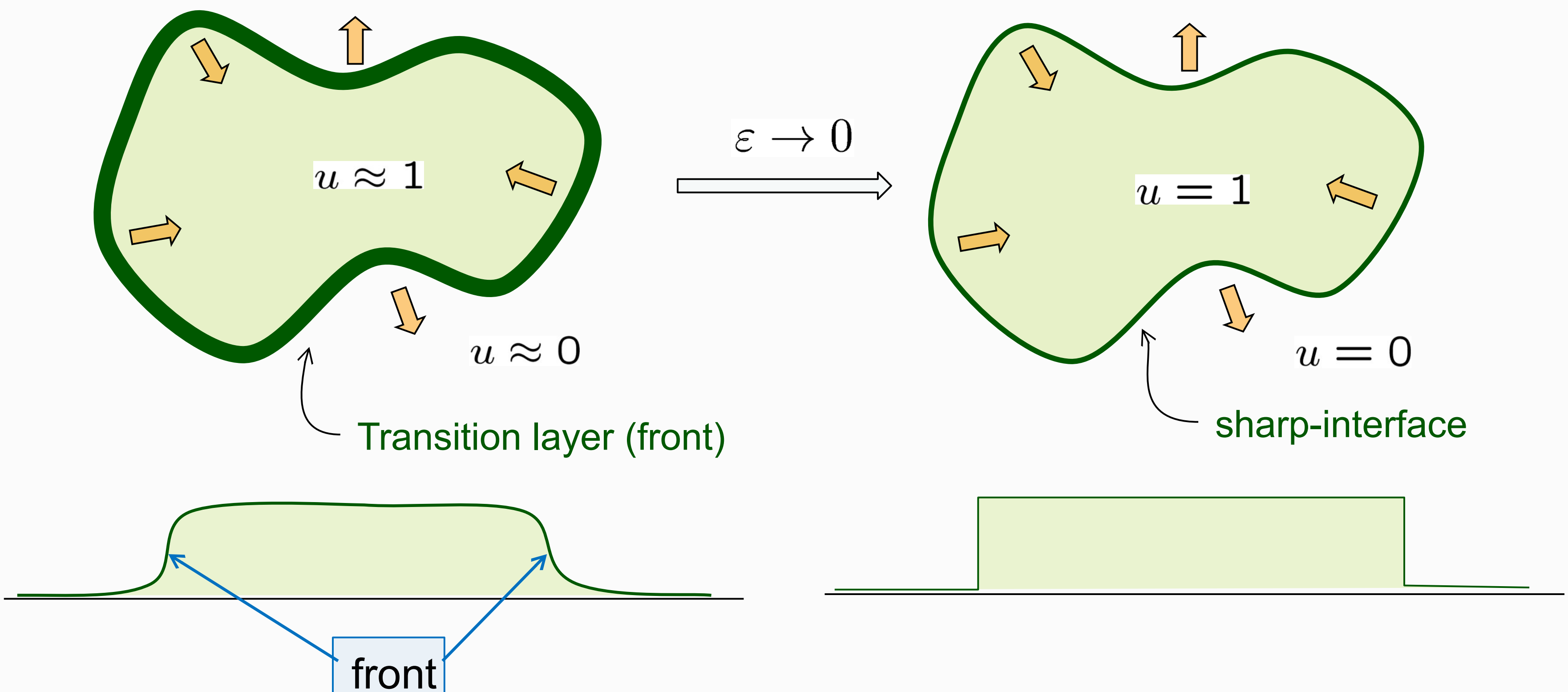
Energy $\boxed{E(u) := \int \left( \frac{1}{2}|\nabla u|^2 + W(u) \right) dx}$

Interfacial energy
+ bulk energy

*Athens 2025*

# More rigorous approach: sharp-interface limit

$$\frac{\partial u}{\partial t} = \Delta u + \frac{1}{\varepsilon^2}\big(f(u) - \varepsilon g(u)\big) \xrightarrow[\text{sharp-interface limit}]{\text{Singular limit}} \text{mean curvature flow}$$




Expample:

$$\frac{\partial u}{\partial t} = \Delta u + \frac{C}{\varepsilon^2} u(1-u)\left(u - \frac{1}{2} - \varepsilon a\right) \Longrightarrow V = \kappa + A$$

# 2. Formulation of the problem

# Planar-front like solution

$$(E)\quad \begin{cases} u_t = \Delta u + f(u), & x \in \Omega := \mathbb{R}^N \setminus K \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

Is there a planar-front like solution that approaches the obstacle K ?

**Theorem 1.** *Assume* $K \subset \{x \in \mathbb{R}^N \mid x_1 \geq 0\}$, *There exists a unique entire solution* $\bar{u}$ *of* (E) *satisfying* $0 < \bar{u} < 1$ $(x \in \overline{\Omega}, t \in \mathbb{R})$ *and*

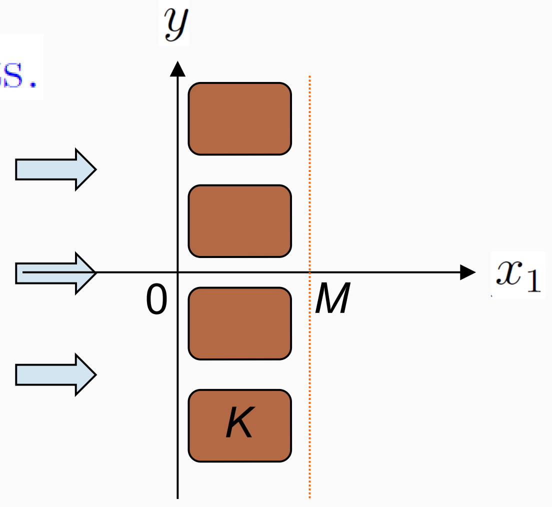
$$\lim_{t \to -\infty} \sup_{x \in \Omega} |\bar{u}(t,x) - \phi(x_1 - ct)| = 0.$$

*This solution satisfies* $\bar{u}_t > 0$ *for all* $x \in \overline{\Omega}$, $t \in \mathbb{R}$.

Since $\bar{u}$ is monotone in $t$, the following limit exists.

$$\boxed{\bar{v}(x) := \lim_{t \to +\infty} \bar{u}(t,x) \quad \text{(limit profile)}}$$

$$(S)\quad \begin{cases} \Delta v + f(v) = 0, & x \in \Omega := \mathbb{R}^N \setminus K \\ \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$



# Limit profile

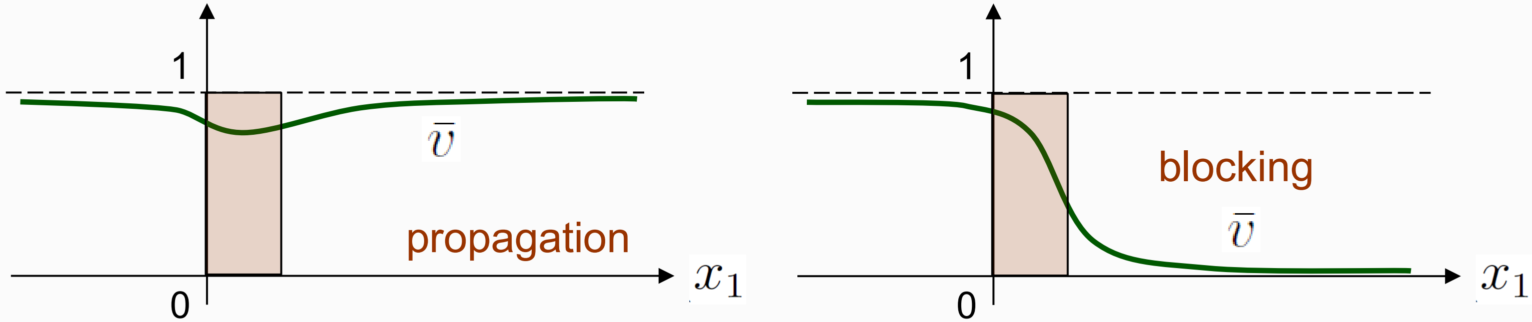
$$(E)\quad \begin{cases} u_t = \Delta u + f(u), & x \in \Omega := \mathbb{R}^N \setminus K \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

$$\boxed{\bar{v}(x) := \lim_{t\to+\infty} \bar{u}(t,x) \quad \text{(limit profile)}}$$

$$\begin{cases} \Delta v + f(v) = 0, & x \in \Omega := \mathbb{R}^N \setminus K \\ \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

$0 < \bar{v} \le 1 \ \text{ in } \ \Omega,$ $\quad \displaystyle\lim_{x_1\to-\infty} \bar{v}(x) = 1.$ $\quad$ $\bar{v}$ is stable from below.

**Def** $\displaystyle\lim_{x_1\to+\infty} \bar{v}(x_1, y) = \begin{cases} 1 & \text{(propagation)}, \\ 0 & \text{(blocking)}. \end{cases}$




**Q** Find conditions for propagation and those for blocking.

# List of main results

- Dichotomy theorem

  The limit profile converges either to 1 or to 0, uniformly regardless of the choice of $K$.

  Method: Liouville type lemma

  The family of blocking walls form a closed set, while that of non-blocking walls form an open set.

- Sufficient conditions for blocking.

  Blocking occurs if the holes are narrow enough.

  Method: variational arguments

- Sufficient conditions for propagation.

  (a) Wall with large holes;
  (b) Small capacity wall;
  (c) Parallel blade wall.

# 3. Propagation / blocking dichotomy

Classification of solution behaviors beyond the wall.

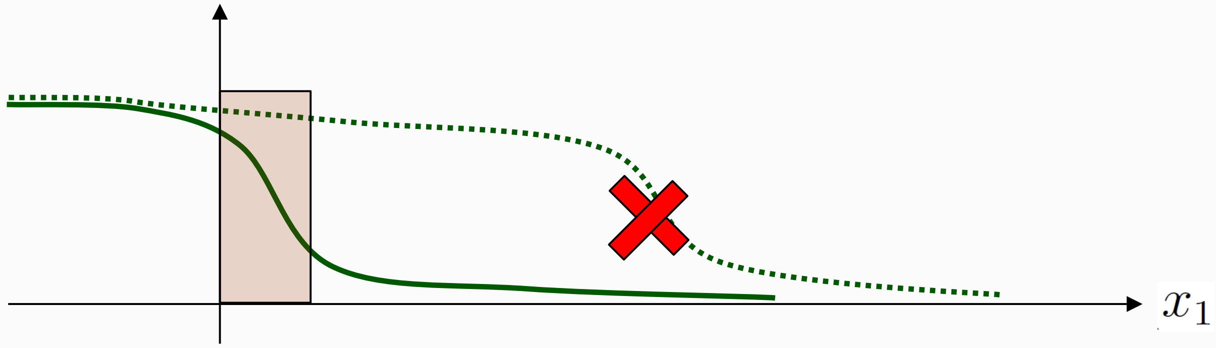
# Propagation / blocking dichotomy

**Theorem 2** (Dichotomy). *One of the following alternatives holds:*

$$\lim_{x_1\to+\infty} \bar{v}(x_1,y)=1 \quad \text{(propagation)}, \qquad \lim_{x_1\to+\infty} \bar{v}(x_1,y)=0 \quad \text{(blocking)}$$

*The above convergence is uniform with respect to* $y\in\mathbb{R}^{N-1}$ *and* $K$ *so long as* $K\subset\{x\in\mathbb{R}^N \mid 0\le x_1\le M\}$.

In particular, there is no blocking profile that converges to 0 too slowly.




Idea of proof: Limiting argument + Liouville type lemma

# Liouville type lemma

[ Y. Liu, K. Wang, J. Wei, K. Wu: Proc. AMS, to appear ]

**Lemma 1.1** (Liouville type lemma). *Let $g:\mathbb{R}\to\mathbb{R}$ be a $C^2$ function whose zeros are all isolated, and let $v(x)$ be a bounded solution of*

$$\Delta v + g(v) = 0 \quad in \ \ \mathbb{R}^N$$

*that is <u>stable</u>. Assume that the one-dimensional equation $w'' + g(w) = 0$ does not have a nonconstant stable solution. Then $v$ is a constant.*

Definition of "stability"

$$\begin{cases} -\Delta\phi - g'(v)\phi = \lambda_R\phi, \ \phi > 0 & \text{in } B_R, \\ \phi = 0 & \text{on } \partial B_R. \end{cases}$$

$$\boxed{\lambda_R \geq 0 \text{ for all } R > 0.}$$

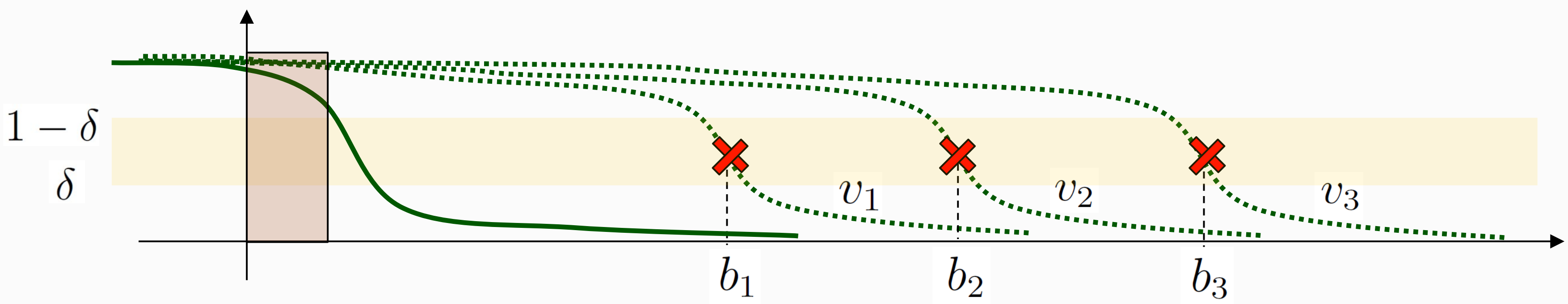
**Corollary 4.** *Let $v$ be a bounded solution of*

$$\Delta v + f(v) = 0 \quad in \ \ \mathbb{R}^N$$

*that is stable. Then either $v = 0$ or $v = 1$.*

## Outline of the proof of Theorem 2

Suppose that there exists a sequence of walls $K_j$ $(j = 1, 2, 3, \ldots)$ such that the limit profiles $v_j$ for $K_j$ do not converge to 1 nor 0 fast enough.




By shifting $v_j$ and taking the limit, we obtain a function $v_\infty$ on $\mathbb{R}^N$ s.t.

$$0 < v_\infty(x) < 1 \quad \text{on} \quad \mathbb{R}^N, \qquad v_\infty \text{ is stable.}$$

But this contradicts the Liouville type lemma. $\square$

### Remark

Stability is preserved by spatial shifts and by limiting procedures.

# Propagation / blocking dichotomy

**Theorem 2** (Dichotomy). *One of the following alternatives holds:*

$$\lim_{x_1\to+\infty} \bar{v}(x_1,y) = 1 \quad \text{(propagation)}, \qquad \lim_{x_1\to+\infty} \bar{v}(x_1,y) = 0 \quad \text{(blocking)}$$

*The above convergence is uniform with respect to $y \in \mathbb{R}^{N-1}$ and $K$ so long as $K \subset \{x \in \mathbb{R}^N \mid 0 \le x_1 \le M\}$.*

**Corollary 3.** *Let $K_1, K_2, K_3, \ldots$ be a sequence of walls satisfying*

$$K_j \subset \{x \in \mathbb{R}^N \mid 0 \le x_1 \le M\} \quad (j = 1, 2, 3, \ldots)$$

*and converging to a wall $K_\infty$ in the Hausdorff distance. If blocking occurs for every $K_j$ $(j = 1, 2, 3, \ldots)$ then the same holds for $K_\infty$.*

$\bar{v}_j$ $(j = 1, 2, 3, \ldots)$ limit profile of $K_j$ $\qquad \bar{v}_j \to \bar{v}_\infty$

$$\Downarrow$$

$$\lim_{x_1\to+\infty} \bar{v}_\infty(x_1,y) = 0$$

# 4. Conditions for blocking

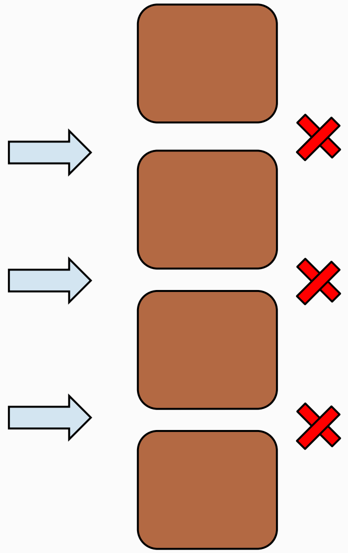
## Geometric obstruction

**Theorem 5.** Assume either of the following:

(K1) $K$ is periodic in $y$.

(K2) The holes are localized in a bounded region.

If the holes are too small, then blocking occurs.

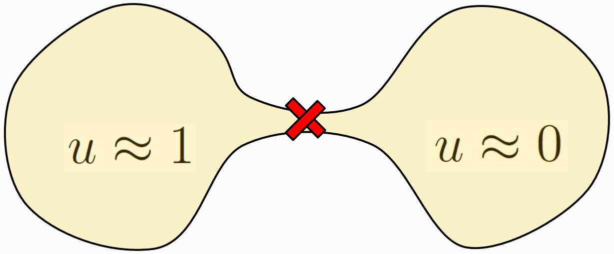


**Idea of proof**

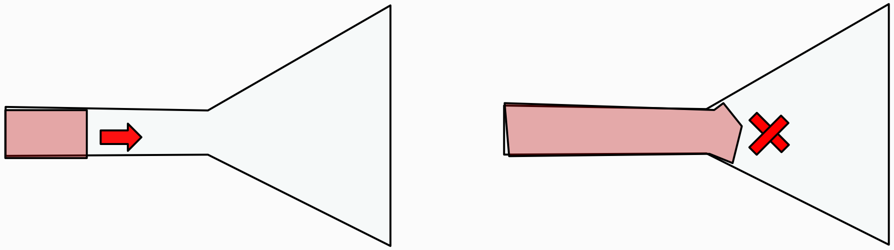
Construction of an upper barrier by variational methods.

**Related earlier works.**

- [M. 1979]
- [Berestycki-Bouhours-Chapuisat 2016]




(M 1979)



(BBC 2016)

Theorem 5. Assume either of the following:

(K1) $K$ is periodic in $y$.

(K2) The holes are localized in a bounded region.

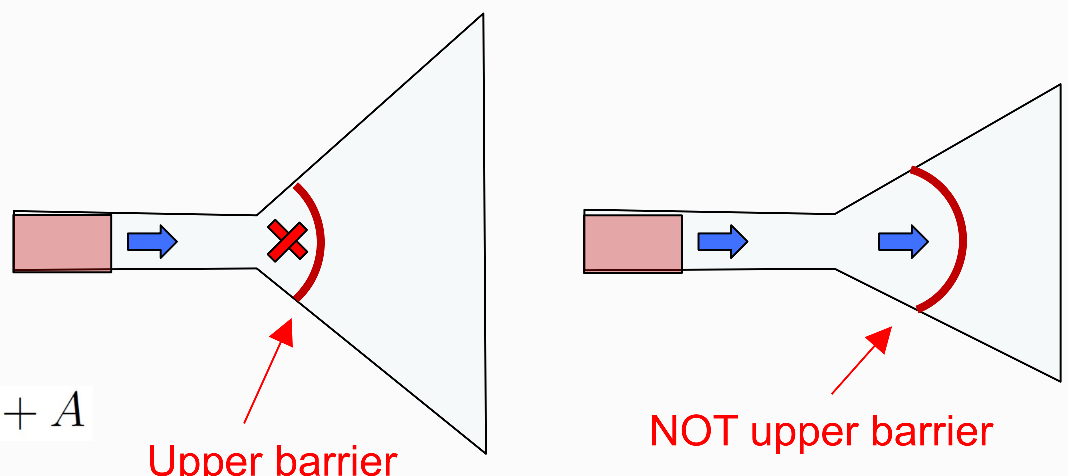
If the holes are too small, then blocking occurs.

Remark

- The assumptions (K1) and (K2) allow us to define an energy functional around the holes. The problem is open without these conditions.
- Whether blocking occurs or not does not simply depend on the size of the holes. As shown in [BBC 2016], blocking occurs if the opening angle is large, but not if the opening angle is small.

This result can be explained intuitively by MMC approximation using a circular arc of radius

$$u_t = \Delta u + f(u) \;\approx\; V = \kappa + A$$



**Theorem 5.** Assume either of the following:

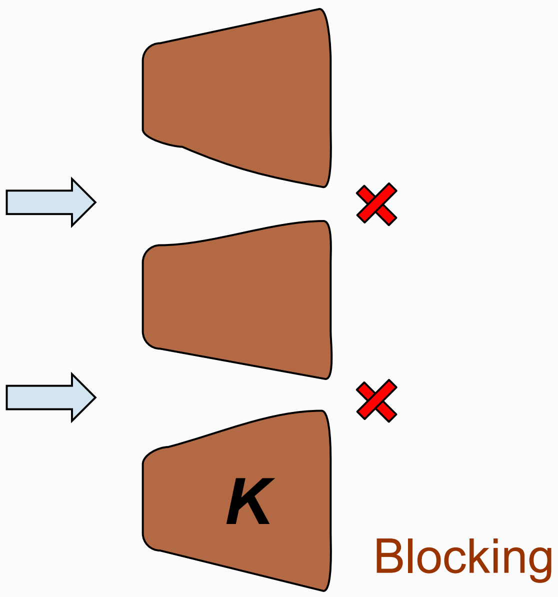
(K1) $K$ is periodic in $y$.

(K2) The holes are localized in a bounded region.

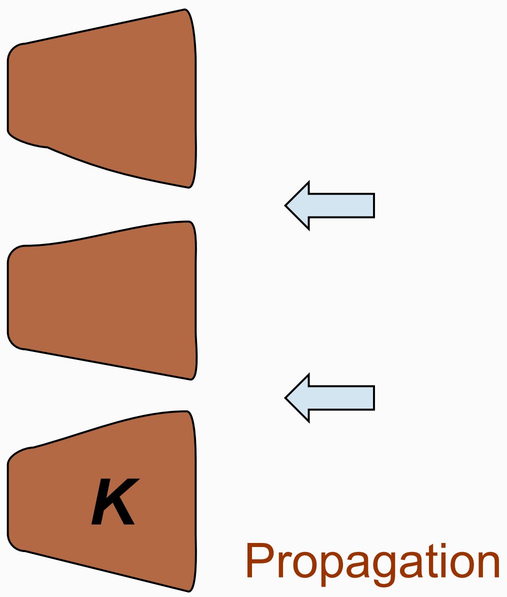
If the holes are too small, then blocking occurs.

One-way blocking

(A) Narrow exit and large opening angles




(B) Narrow entrance but wide exit, and small opening angles.



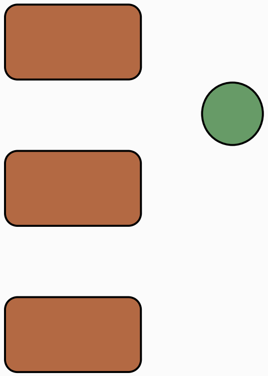
# 5. Sufficient conditions for propagation

Three types of walls

# Walls that allow propagation

## (a) Wall with large holes

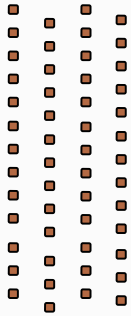
A ball of a critical radius $R_0$ can pass through one of the holes, where $R_0$ is to be specified later.



## (b) Small capacity wall

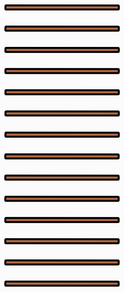
$K$ is close to a set of capacity 0 in Hausdorff distance.

($K_\varepsilon$ is in the $\varepsilon$ neighborhood of a zero capacity set $K_0$.)



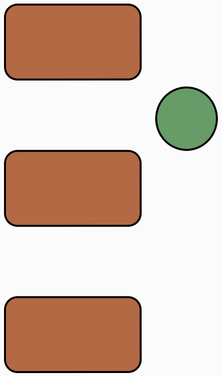
## (c) Parallel-blade wall

$K$ consists of thin panels parallel to the $x_1$ axis. More precisely, $K_0$ is a locally finite union of hypersurfaces parallel to the $x_1$ axis and let $K_\varepsilon$ converge to $K_0$ in a certain sense.



# Walls that allow propagation

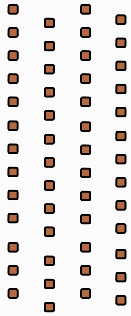
## (a) Wall with large holes



A ball of a critical radius $R_0$ can pass through one of the holes, where $R_0$ is to be specified later.

Method: comparison principle and sweeping argument

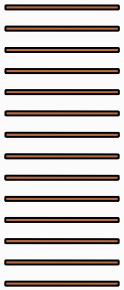
## (b) Small capacity wall



$K$ is close to a set of capacity 0 in Hausdorff distance.

Method: limiting argument, removal singularity, and dichotomy theorem (Cor.2).

## (c) Parallel-blade wall



$K$ consists of thin panels parallel to the $x_1$ axis.

Method: sweeping method by "quasi-subsolutions" and relative Poincaré inequality

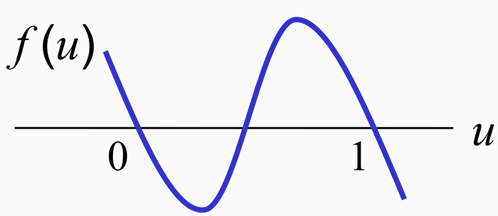
# 1. Wall with large holes

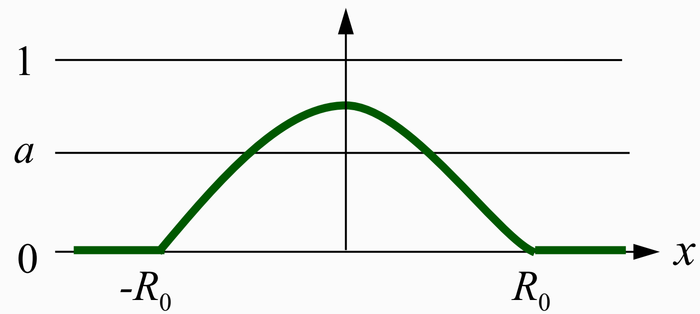
Theorem 6. If a ball of radius $R_0$ can pass through one of the holes of the wall without touching $K$, then propagation occurs .

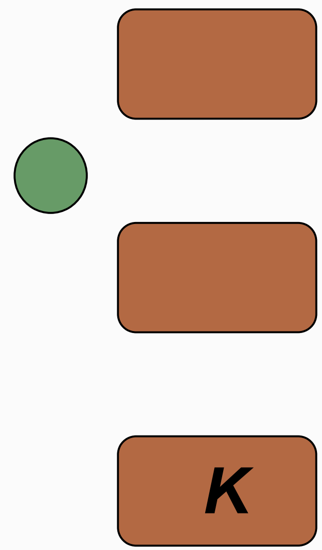
Let $\Psi(x) \geq 0$ be a compactly supported radial subsolution:

$$\Delta\Psi + f(\Psi) = 0 \ (|x| < R_0), \quad \Psi(x) = 0 \ (|x| \geq R_0)$$

satisfying $\alpha < \Psi(0) = \max\Psi < 1$ (Aronson-Weinberger '78).




$$\Psi^P(x) = \begin{cases} \Psi(x-P) & \text{if } |x-P| \leq R_0, \\ 0 & \text{otherwise.} \end{cases}$$

Strong maximum principle + sweeping argument

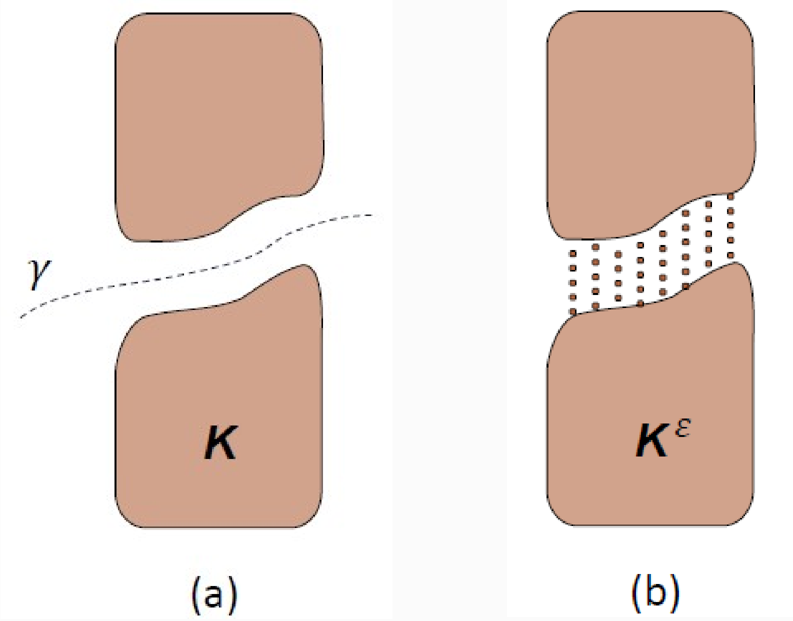
Move the position of P continuously without hitting $K$

# 2. Small capacity wall

**Theorem 7** (Small-capacity wall). *Let $K^\varepsilon$ $(0 < \varepsilon \le \varepsilon_0)$ be a family of walls that is periodic in $y \in \mathbb{R}^{N-1}$ and satisfies*

$$K^\varepsilon \subset \{x \in \mathbb{R}^N \mid 0 \le x_1 \le M\}, \quad \limsup_{\varepsilon \to 0} K^\varepsilon \subset K_0 \cup K_1,$$

*where $K_0$ is a closed set of capacity $0$, $K_1$ is a wall with large holes or $K_1 = \emptyset$. Then for all sufficiently small $\varepsilon > 0$, propagation occurs for $K^\varepsilon$.*




(a) Wall with a large hole.

(b) Wall with a large hole that is filled with debris of small capacity.

# 2. Small capacity wall

**Theorem 7** (Small-capacity wall). *Let $K^\varepsilon$ $(0 < \varepsilon \le \varepsilon_0)$ be a family of walls that is periodic in $y \in \mathbb{R}^{N-1}$ and satisfies*

$$K^\varepsilon \subset \{x \in \mathbb{R}^N \mid 0 \le x_1 \le M\}, \quad \limsup_{\varepsilon \to 0} K^\varepsilon \subset K_0 \cup K_1,$$

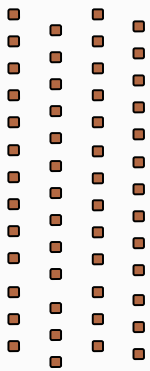
*where $K_0$ is a closed set of capacity 0, $K_1$ is a wall with large holes or $K_1 = \emptyset$. Then for all sufficiently small $\varepsilon > 0$, propagation occurs for $K^\varepsilon$.*

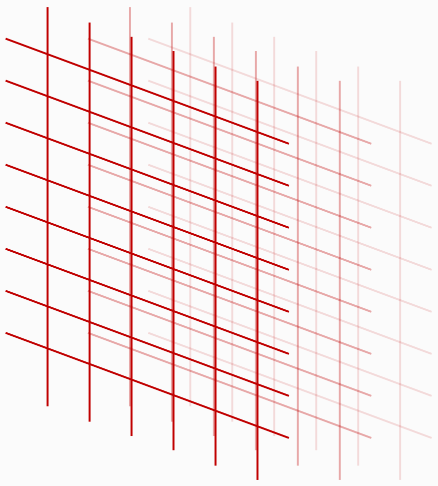
## Sufficient conditions for capacity 0

- Hausdorff dimension of $K_0 < N - 2$.
- $(N-2)$ dimensional rectifiable manifold.

## Examples of $K_0$

- Discrete set (N ≥ 2) → "Debris wall"
- Locally finite union of curves (N = 3) → "Filament wall"





Athens 2025

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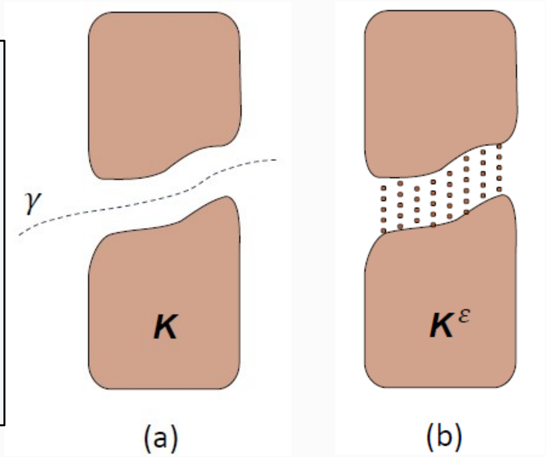



Idea of proof: Limiting argument + Removable singularity

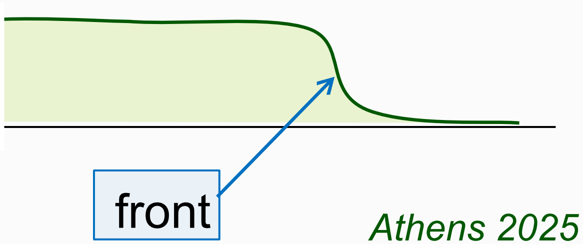
+ Dichotomy theorem

(Corollary 3)

**Corollary 3.** The limit of a sequence of blocking walls is a blocking wall.

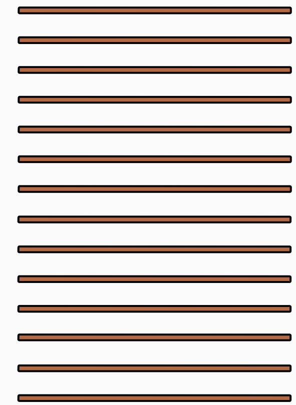
Interpretation of the result

Since the front has positive thickness, it is not very sensitive to debris of small capacity.

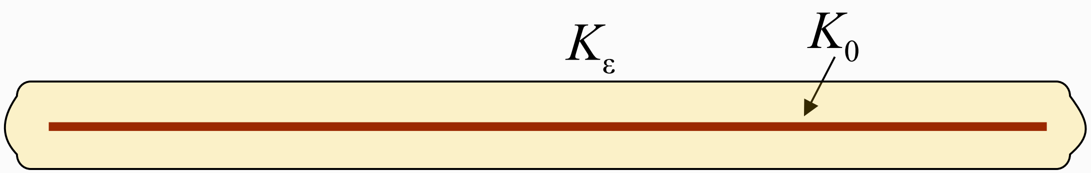


# 3. Parallel-blade wall

is periodic in and consists of thin panels parallel to the $x_1$ axis.



More precisely, lies in the neighborhood of which is an N-1 dimensional set parallel to $x_1$ axis.




$$\int_{\partial K^\varepsilon \cap \Delta_{\mathcal{P}}} |\nu \cdot \boldsymbol{e}_1| \, dS_x \leq \varepsilon_1$$

Note: $K_0$ has positive capacity as codim $K_0$ =1

In 3D, a honeycomb wall is also an example.

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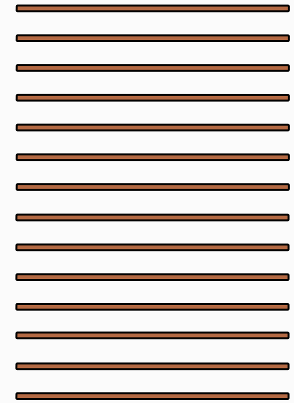

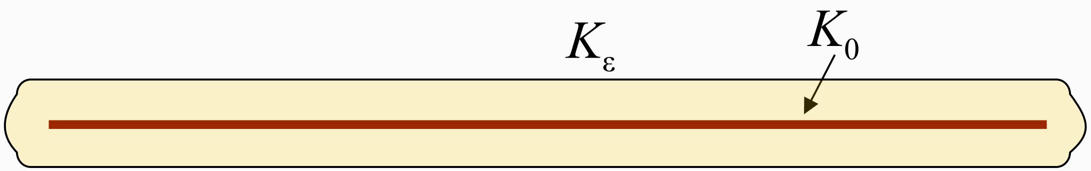


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**Theorem 8** (Parallel-blade wall). *Let $K^\varepsilon$ $(0 < \varepsilon \leq \varepsilon_0)$ be a family of $\mathcal{P}$-periodic parallel-blade walls converging to $K_0 \subset [0, M] \times \Sigma$ as $\varepsilon \to 0$, Then for all sufficiently small $\varepsilon > 0$, propagation occurs for $K^\varepsilon$.*

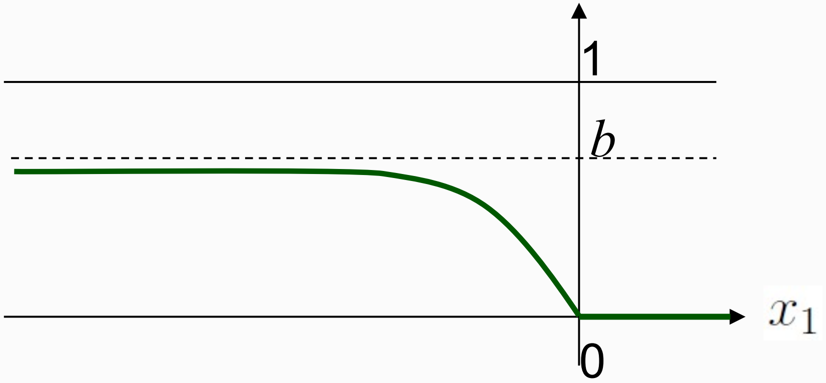
Method: sweeping method by “quasi-subsolutions” and relative Poincaré inequality

**Theorem 8** (Parallel-blade wall). *Let $K^\varepsilon$ $(0 < \varepsilon \le \varepsilon_0)$ be a family of $\mathcal{P}$-periodic parallel-blade walls converging to $K_0 \subset [0, M] \times \Sigma$ as $\varepsilon \to 0$, Then for all sufficiently small $\varepsilon > 0$, propagation occurs for $K^\varepsilon$.*

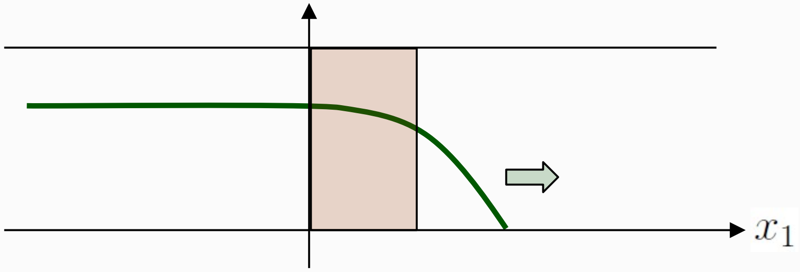
<u>Strategy of proof</u>: Sweeping argument by quasi-subsolutions.

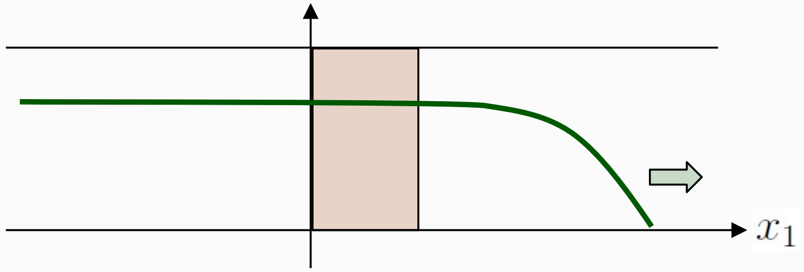
**Step 1** Define a function on by

$$\begin{cases} \rho'' + f(\rho) = \delta & \text{if } \ x_1 < 0 \\ \rho > 0, \quad \rho' < 0 & \text{if } \ x_1 < 0 \\ \rho(x_1) = 0 & \text{if } \ x_1 \ge 0 \\ \rho(x_1) \to b & \text{as } \ x_1 \to -\infty. \end{cases}$$




**Step 2** Define $\rho^\lambda(x) := \rho(x_1 - \lambda)$. Then increase $\lambda$ continuously.



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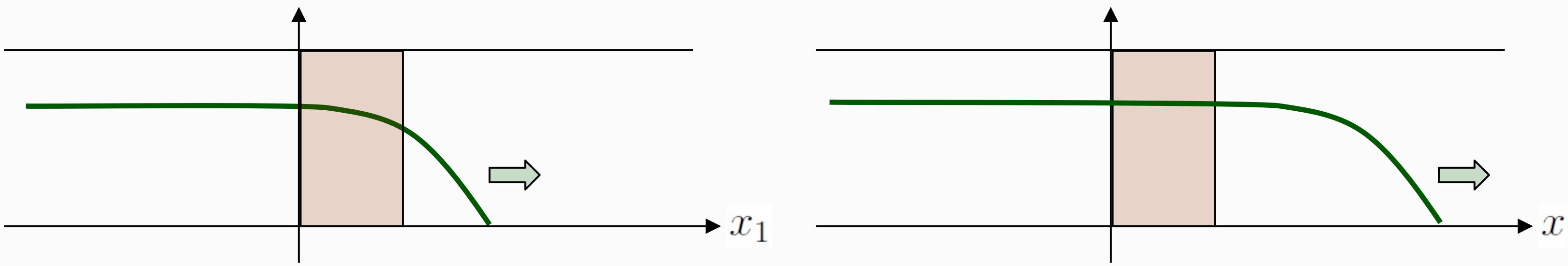
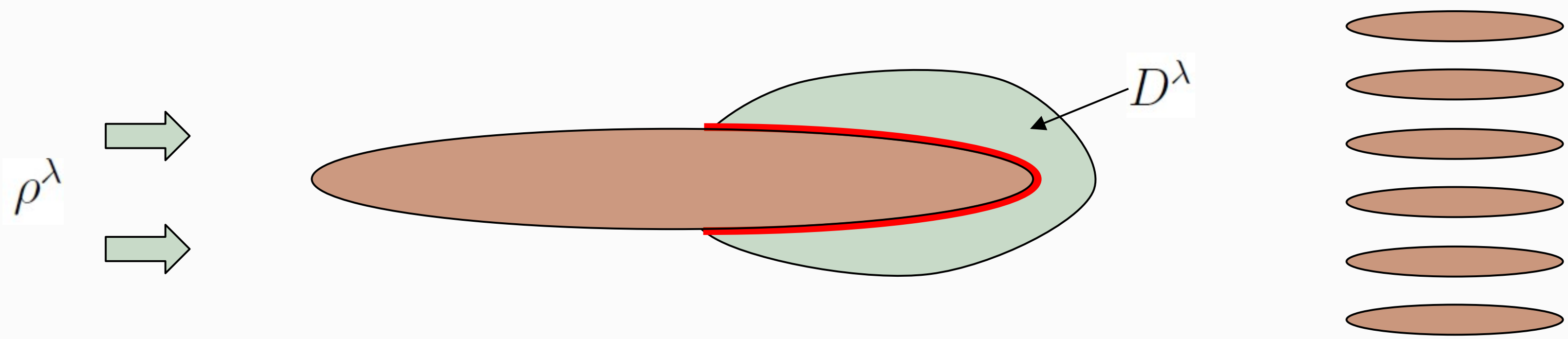



**Remark** is not entirely a subsolution, since it fails to satisfy the boundary condition on part of .




**Step 3** Define $D^\lambda := \{x \in \overline{\Omega} \mid \rho^\lambda(x) > \bar{v}^\varepsilon(x)\} \cap \boxed{\Delta_{\mathcal{P}}}$ ← unit periodicity cell

Relative Poincare inequality ⇨ $\boxed{|D^\lambda| \leq \nu, \quad \textit{for all} \quad \lambda \in \mathbb{R}.}$

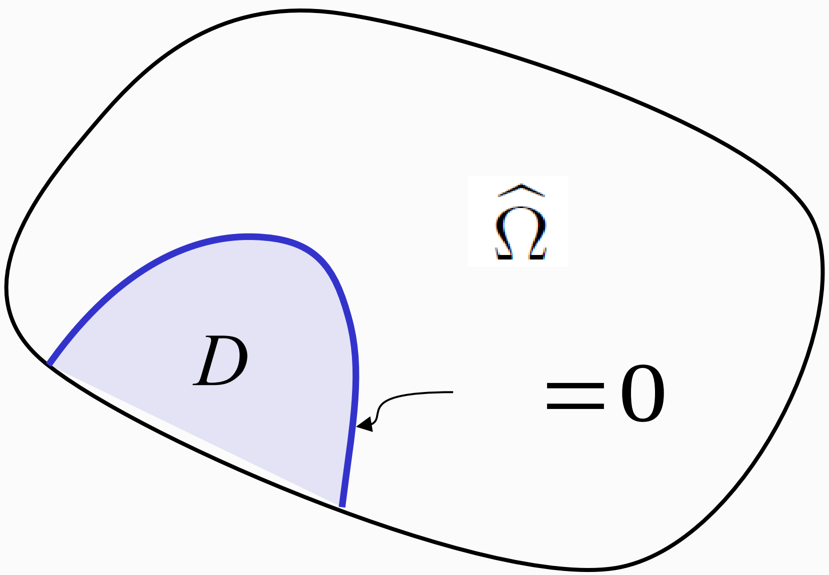
This implis propagation, since otherwise as .

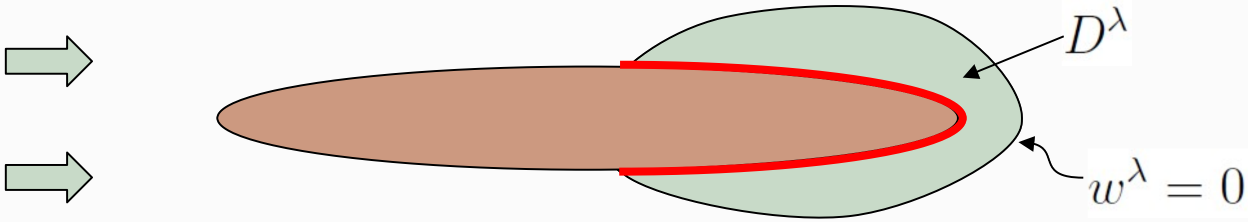
# Relative Poincaré inequality

G. Buttazzo and B. Velichkov, *The spectral drop problem*, Contemporary Math. **666** (2016), pp. 111–135.

**Proposition 1.3.** *Let $\widehat{\Omega}$ be a domain in $\mathbb{R}^N$ and let $\eta_0$ be a real number with $0 < \eta_0 < |\widehat{\Omega}|$. Then there exists a constant $C > 0$ depending only on $\widehat{\Omega}$ and $\eta_0$ such that, for any open set $D \subset \widehat{\Omega}$ satisfying $|D| \leq \eta_0$ and any $w \in H^1(\widehat{\Omega}) \cap C(\widehat{\Omega})$ such that $w = 0$ in $\widehat{\Omega} \setminus D$, the following holds:*

$$\int_D |\nabla w|^2 dx \geq C|D|^{-\frac{2}{N}} \int_D w^2 dx.$$






$\boxed{\text{Step 3}}$ Define $D^\lambda := \{x \in \overline{\Omega} \mid \rho^\lambda(x) > \bar{v}^\varepsilon(x)\} \cap \Delta_{\mathcal{P}}$

We show that $\boxed{|D^\lambda| \le \nu, \quad \textit{for all} \quad \lambda \in \mathbb{R}.}$

Let $w^\lambda := \rho^\lambda - \bar{v}^\varepsilon.$

$$\begin{cases} \Delta w^\lambda + f'(\xi(x))w^\lambda = \delta, & \text{in } \Omega \\ w^\lambda > 0 & \text{in } D^\lambda, \\ w^\lambda = 0, & \text{on } \partial D^\lambda \cap \Omega \end{cases}$$

$\Rightarrow$ $|D^\lambda| \le \delta^{-2} L_f \left( - \int_{\widehat{D}^\lambda} |\nabla w|^2 + L_f \int_{\widehat{D}^\lambda} w^2 \right) + C_1(\varepsilon + \varepsilon_1)$

$\Rightarrow$ $|D^\lambda| \le \delta^{-2} L_f \left( -C|\widehat{D}^\lambda|^{-2/N} + L_f \right) \int_{\widehat{D}^\lambda} w^2 + C_1(\varepsilon + \varepsilon_1)$ relative Poincaré inequality

$\Rightarrow$ $\exists \eta_1 > 0 \;\text{ such that }\; |D^\lambda| \le \eta_1 \implies |D^\lambda| \le C_1(\varepsilon + \varepsilon_1)$

$\Rightarrow$ $\boxed{|D^\lambda| \le \nu, \quad \textit{for all} \quad \lambda \in \mathbb{R}.}$ $\Rightarrow$ Propagation

## Summary:

We discussed whether or not a planar bistable front can propagate through a perforated wall.

1. Classification of general behavior:

   Dichotomy theorem (via Liouville type lemma)

   Cor. the limit of blocking walls is again blocking.

2. A sufficient condition for blocking: (narrow holes)

   Method: construction of an upper barrier via a variational argument

3. Three sufficient conditions for propagation:

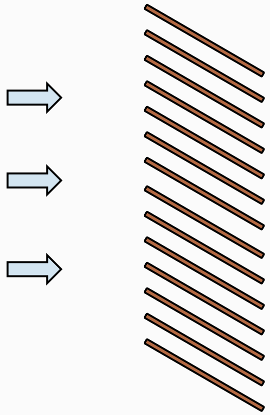
   (a) large-hole walls (b) small-capacity walls (c) parallel-blade walls.

   Method: (a) sweeping method, (b) removable singularity theory + dichotomy thm, (c) sweeping by quasi-subsolutions.

## Open questions:

1. Tilted thin panels:

   What if the panels are tilted ?

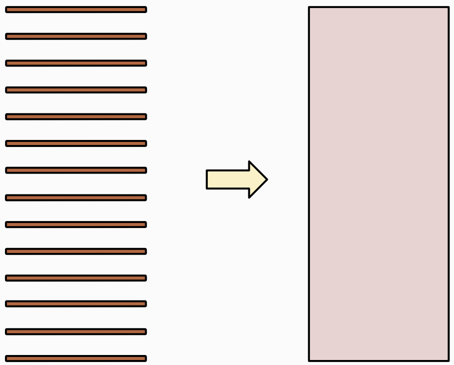


2. Proof of blocking for non-periodic walls

   Is there a way to apply variational arguments?

3. Homogenization problem:

   What if the size of the holes and
   the distance between adjacent holes
   both go to zero simultaneously?



Thank you!