Singular harmonic maps into singular one-dimensional manifold

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Partially based on the thesis of Mehdi Trense

Let Σ be a compact connected one-dimensional rectifiable set, such that there exists a finite set

$$\Theta_\star = \{\upsilon_1, \dots, \upsilon_\ell\} \subset \Sigma,$$

Such that

$$\Sigma' = \Sigma \setminus \Theta_\star \text{ is smooth.}$$

If $y_0 \notin \Theta_\star$, $\exists$ an open neighborhood of $\mathscr{I}_0$ of $y_0 \in \Sigma'$, $a > 0$, and

$$\varphi : \mathscr{I}_0 \to [-a, a] \text{ with } \varphi(0) = y_0,$$

an **arc-length parametrization** of $\mathscr{I}_0$
(*assuming some given some metric on* Σ). Let $\vec{\tau}_{y_0}$ denotes an unit tangent vector to Σ at y_0 such that,

$$\frac{d\varphi^{-1}}{ds}(0) = \vec{\tau}_{y_0}. \tag{1}$$

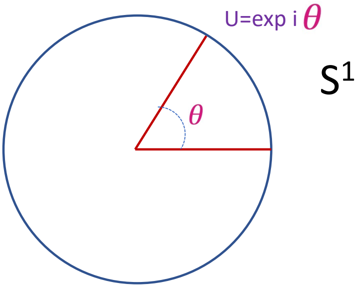
The regular case : $\Theta_\star = \emptyset$

In the case Σ is smooth, one may assume, by compactness and connectedness that

$$\Sigma = \mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2, x^2 + y^2 = 1\}$$

and the parametrization is given by

$$u = \exp(i\theta), \theta \in \mathbb{R}.$$



Singularities for Σ

The allowed singularities for Σ have a simple structure. Given a singularity υ_j, we assume that there exists a neighborhood $\mathcal{O}$ of υ_j, and a finite number $q(j) \geq 3$ of smooth curves $\mathcal{C}_{j,1}, \ldots, \mathcal{C}_{j,q(j)}$ such that

$$\begin{cases} \Sigma \cap \mathcal{O} = \bigcup_{k=1}^{q(j)} \mathcal{C}_{j,k} \text{ and} \\ \mathcal{C}_{j,k} \cap \mathcal{C}_{j,k'} = \emptyset, \text{ if } k \neq k' \\ q(j) \geq 3. \end{cases} \tag{2}$$

Remark

The case $q(j) = 2$ *does not give rise to a singularity*. Indeed, using appropriate charts, the set Σ may be given a smooth structure in a neighborhood of such a point.

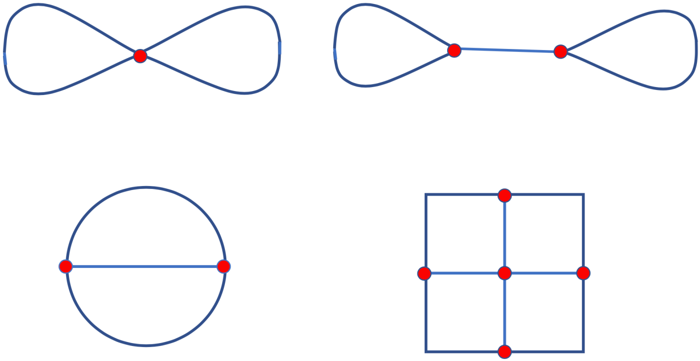


Figure: *One-dimensional manifolds with one, two or four singularities*

This set has the shape of the symbol ∞, or of the figure 8: It is the union of two loops glued at one common point, the length of each of the two loops is exactly one. The exact geometric realization of the set, is of little relevance. Set

$$\Sigma_\infty \equiv \mathbb{S}^+ \cup \mathbb{S}^-, (\text{ often written as } \mathbb{S}^+ \vee \mathbb{S}^-) \tag{3}$$

where $\mathbb{S}^\pm$ denote the circles

$$\mathbb{S}^\pm = \mathbb{S}(\pm(2\pi)^{-1}, (2\pi)^{-1}), \text{ so that } \mathbb{S}^+ \cap \mathbb{S}^- = \{0\}$$

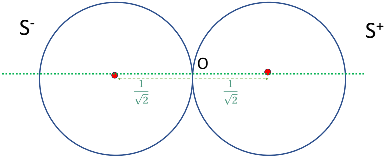
so that the length of $\mathbb{S}^\pm$ is $|\mathbb{S}^\pm| = 1$.

Here, for $r > 0$ and $a \in \mathbb{C}$ the set $\mathbb{S}(a,r)$ denotes the circle of radius r centered at the point a, i. e.

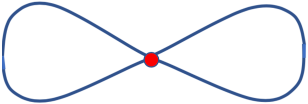
$$\mathbb{S}^1(a,r) = \{z \in \mathbb{C}, |z-a| = r\}. \tag{4}$$

Σ_∞ is a metric space: The distance between two points on $\mathbb{S}^+$ (resp. $\mathbb{S}^-$) is given by the arc length, whereas for the distance between a point on $\mathbb{S}^-$ and a point on $\mathbb{S}^+$ one adds the distances of each of these points to the point 0, computed using arc-length.

Two representations of Σ_∞



$\simeq$



Recall that the fondamental group $\pi_1(\Sigma)$ of a set Σ is given by

$$\pi_1(\Sigma) = C^0(\mathbb{S}^1, \Sigma) / \textbf{ homotopy}$$

equipped with a suitable group structure $+$.

The case $\Sigma = \mathbb{S}^1$. Here we have $\Sigma(S^1) \sim \mathbb{Z}$. The generator α corresponds to the homotopy class of the identity map from $\mathbb{S}^1$ to $\mathbb{S}^1$. and stands for 1 in $\mathbb{Z}$ (this number is called the degree). A map with degree $d \in \mathbb{Z}$ is given by

$$f_d(\exp(i\theta)) = \exp(id\theta).$$

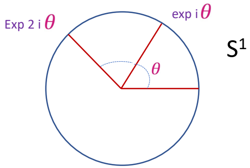
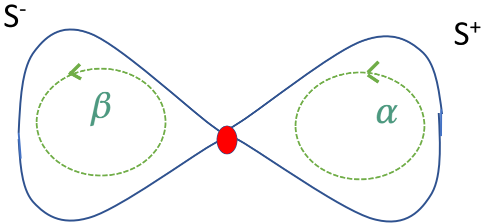


Figure: *The case* $d = 2$

The group $\pi_1(\Sigma_\infty \sim \mathbb{S}^+ \vee \mathbb{S}^-)$ has two generators α and β: α (resp. β) corresponds to the homotopy class of $f_\alpha : \mathbb{S}^1 \to \Sigma_\infty$ (resp. f_β) defined from by

$$f_\alpha(\exp i\theta) = \frac{1}{2\pi}(\exp i\theta - 1) \left(\text{resp. } f_\beta(\exp i\theta) = \frac{1}{2\pi}(\exp i\theta + 1)\right), \theta \in \mathbb{R}, \qquad (5)$$

so that f_α (resp. f_β) maps $\mathbb{S}^-$ (resp. $\mathbb{S}^+$) with degree one.



The generators α and β **do not commute**, any $\omega \in \pi_1(\Sigma_\infty)$ is a a "word" made of the "letters" α's, α^{-1}'s, β's, and β^{-1}'s: A word ω is an element of

$$\mathfrak{D} = \bigcup_{k=0}^{\infty} \mathfrak{A}^k, \quad \text{where } \mathfrak{A} = \{\alpha, \alpha^{-1}, \beta, \beta^{-1}\} \text{ is the alphabet.}$$

If $\omega = (\omega_1, \dots \omega_k) \in \mathfrak{D}$, the corresponding element in $\pi_1(\Sigma_\infty)$ is given by

$$T(\omega) = \omega_1 \omega_2 \dots \omega_k.$$

The smallest number of letters of $\mathfrak{A}$ necessary to write ω is the length of ω, $|\omega|$. In view of invariance by rotation of the circle circular permutation yield the same element. For instance, for the commutator γ defined by

$$\gamma = \alpha\beta\alpha^{-1}\beta^{-1}, \text{ we have } \gamma = \beta\alpha^{-1}\beta^{-1}\alpha = \alpha^{-1}\beta^{-1}\alpha\beta = \beta^{-1}\alpha\beta\alpha^{-1},$$

and

$$\gamma^{-1} = \beta\alpha\beta^{-1}\alpha^{-1} = \alpha\beta^{-1}\alpha^{-1}\beta = \beta^{-1}\alpha^{-1}\beta\alpha.$$

A word is said to be **closed** if it contains as much α's as α^{-1}'s, as much β's as β^{-1}'s. γ and γ^1 are closed.

The "Télécran" (Etch a Sketch")

A toy invented in 1960 by André Chassagnes.

Figure: *The "Télécran". The generators* α *and* β *are the two white buttons at the bottom*

In 1998, inducted into the National Toy Hall of Fame, Rochester, New York.
In 2003, named one of the 100 most memorable toys of the 20th century by the Toy Industry Association.

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$. Consider the Lipschitz projections $\Pi^\pm$ from Σ_∞ to $\mathbb{S}^1$

$$\Pi^\pm(w) = \sqrt{2\pi}(w \pm 1) \text{ if } w \in \mathbb{S}^\pm \text{ and } \Pi^\pm(w) = \sqrt{2\pi} \text{ otherwise, i.e if } w \in \mathbb{S}^\mp.$$

Given u in $C^0(\Omega, \infty)$, we construct U from Ω to $\mathbb{S}^1 \times \mathbb{S}^1$ setting

$$U = (\Pi^+ \circ u, \Pi^- \circ u) \equiv (u^+, u^-) \in C^0(\Omega, \mathbb{S}^1 \times \mathbb{S}^1).$$

If there exists a continuous lifting $\phi^\pm \in C^0(\Omega, \mathbb{R})$ of $u^\pm$ so that

$$u^\pm = \exp i\phi^\pm \text{ on } \Omega,$$

then, we introduce a the "Telecran" map $\mathrm{T_{elc}}(u) \in C^0(\Omega, \mathbb{R}^2)$ defined by

$$\mathrm{T_{elc}}(u)(x) = \frac{1}{2\pi}(\phi^+(x), \phi^-(x)), \quad \text{for } x \in \Omega.$$

$\Omega = \mathbb{S}^1$, $u \sim \gamma = \alpha\beta\alpha^{-1}\beta^{-1}$, then image of Φ is a square.

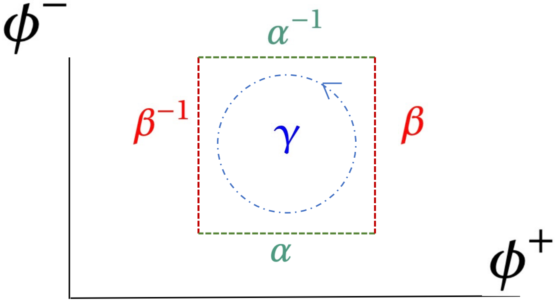


Figure: *The image by the Telecran map* $\mathrm{T}_{\mathrm{elc}}$ *of* γ

$u \sim= \alpha^{-1}\alpha^{-1}\beta^{-1}\alpha\beta^{-1}\beta^{-1}\alpha\beta\alpha\beta\alpha^{-1}\beta,$

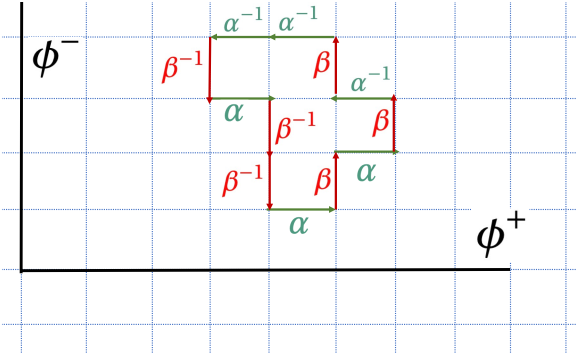


Figure: *The image by* $\mathrm{T}_{\mathrm{elc}}$ *of* u

Let $\mathscr{S}$ be a **smooth** Riemannian manifold. Let : $v : \Omega \to \mathscr{S}$ be a regular map, where $\Omega \subset \mathbb{R}^N$ is open. The map v is said to be a **harmonic map** if i and only if it is locally a critical point on the **Dirichlet Energy**

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2(x)\mathrm{d}x, \quad \text{for } u : \Omega \to \mathscr{S}.$$

If $\mathscr{S}$ is isometrical embedded in some $\mathbb{R}^k$, this is equivalent to

$$\Delta v(x) \perp T_{v(x)}\mathscr{S}, \forall x \in \Omega. \tag{6}$$

If $\dim \mathscr{S} = 1$ and φ est local constant speed parametrization of $\mathscr{S}$ (e.g $y = \exp i\theta$ if $\mathscr{S} = \mathbb{S}^1$), then condition (6) writes

$$\Delta(\varphi \circ v)(x) = 0, \forall x \in \Omega. \tag{7}$$

Harmonic maps may be obtained by minimizing the energy or Min-max procedures.

Assume $\dim\Omega = 2$. Given $u \to \mathscr{S}$, the Hopf differential $\omega(u)$ is the function defined by

$$\omega(u) = |\frac{\partial u}{\partial x_1}|^2 - |\frac{\partial u}{\partial x_2}|^2 - 2i\frac{\partial u}{\partial x_1}\cdot\frac{\partial u}{\partial x_2}$$

The Hopf differential vanishes for conformal maps. For harmonic maps, the Hopf differential is holomorphic

$$\frac{\partial\omega(u)}{\partial\bar{z}} = 0 \text{ on } \Omega.$$

In the case $\mathscr{S}$ is one-dimensional, and $\tau\circ v$ is possibly local constant speed 1 parametrization (e.g. $\tau = \varphi$, if $\mathscr{S} = \mathbb{S}^1$ and $u = \exp i\theta(u(x))$)

$$\omega(u) = \Lambda(u)^2, \text{ with } \Lambda(u) = \frac{\partial(\tau\circ u)}{\partial x_1} - i\frac{\partial(\tau\circ u)}{\partial x_2} = \pm\left[\frac{\partial u}{\partial x_1} - i\frac{\partial u}{\partial x_2}\right]\cdot\vec{e}_{u(x)}.$$

Notice that

$$|\Lambda(u)| = |\nabla u| \text{ and } \Lambda(u) = \text{"}\sqrt{\omega(u)}\text{"}.$$

The notion of Dirichlet energy and minimizing harmonic maps was extended to the case the target Σ is a metric space :

- has constant dimension
- is non-positively curved (NPC),

thanks to works by Gromov-Schoen. In particular Σ_∞ is a (NPC) metric space. The Dirichlet energy of u is given by the formula

$$\begin{cases} E(u) = \lim_{\varepsilon \to 0} \varepsilon^{-N} \int_{M_\varepsilon} \dfrac{d(u(x),u(y))^2}{|x-y|^2} \mathrm{d}x\mathrm{d}y, \\ M_\varepsilon = \{(x,y) \in \Omega \times \Omega, |x-y| \geq \varepsilon\}. \end{cases} \tag{8}$$

They proved the existence and uniqueness of minimizing harmonic maps as well as Lipschitz regularity. Moreover, the Hopf differential is still well-defined and holomorphic:

$$\begin{cases} \omega(u) = |\dfrac{\partial u}{\partial x_1}|^2 - |\dfrac{\partial u}{\partial x_2}|^2 - 2i \dfrac{\partial u}{\partial x_1} \cdot \dfrac{\partial u}{\partial x_2} \\ \dfrac{\partial \omega(u)}{\partial \overline{z}} = 0 \text{ on } \Omega. \end{cases}$$

Maps from the punctured domain $\Omega_\star$ to Σ

Let $a_1, \ldots, a_\ell$ be ℓ distinct points in $\mathbb{R}^2$, consider the punctured domain

$$\Omega_\star = \mathbb{R}^2 \setminus \bigcup_{i=1}^{\ell} \{a_i\}, \text{ and maps } u \in C^0(\Omega_\star, \Sigma), \quad (\Sigma = \mathbb{S}^1 \text{ or } \Sigma = \Sigma_\infty).$$

For u in $C^0(\Omega_*, \Sigma)$, we assign to a singularity a_i the homotopy class $[\![u, a_i]\!]$ of u restricted to any small circle $\mathbb{S}(a_i, r)$, and define its homotopy class at infinity denoted $[\![u]\!]_\infty$ and as the homotopy class of u restricted to any circle $\mathbb{S}(0, R)$ for large $R > 0$. For given $\delta_1, \ldots, \delta_\ell, \delta_\infty \in \pi_1(\Sigma)$,

$$Y^0 = \left\{ u \in C^0(\Omega_*, \Sigma), \text{ such that } [\![u, a_i]\!] = \delta_i, i = 1, \ldots \ell, \text{ and } [\![u]\!]_\infty = \delta_\infty \right\}.$$

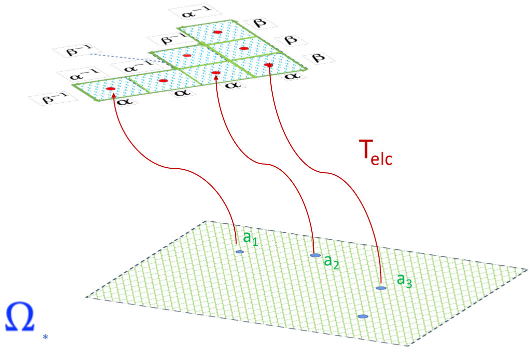
If we want $Y^0 \neq \emptyset$, when compatibilty conditions are required:

If $\Sigma = \mathbb{S}^1$ then $\delta_i \in \mathbb{Z}$ and the compatibility condition is simply

$$\sum_{i=1}^{\ell} \delta_i = \delta_\infty.$$

We choose $\delta_i = [\![u, a_i]\!] = \gamma = \alpha\beta\alpha^{-1}\beta^{-1}$ then the compatibility condition, i.e. the condition for Y^0 not being empty reads as

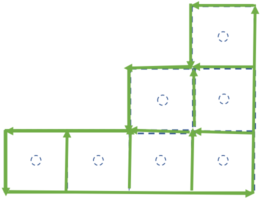
$$\ell = \text{ number of squares bounded by } \mathrm{T_{elc}}([\![u]\!]_\infty).$$



As a remark, we observe that, if $u \in Y_0$, then

$$\det(\nabla \mathrm{T}_{\mathrm{elc}}(u)) = \sum_{i=1}^{\ell} \delta_{a_i}.$$

The image of $\Omega_\star$ by $\mathrm{T}_{\mathrm{elc}}(u)$ is included in the union of the squares.



We introduce a small parameter $\varepsilon > 0$ and consider the set $\Omega_\varepsilon \subset \mathbb{R}^2$

$$\Omega_\varepsilon = B(0, \frac{1}{\varepsilon}) \setminus \bigcup_{i=1}^{\ell} B(a_i, \varepsilon),$$

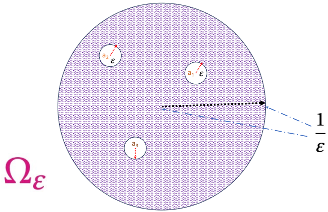
and, for $u : \Omega_\star \to \Sigma$, the "desingularized" energy E_ε defined by $E_\varepsilon(u, \Omega_\varepsilon)$, i.e.

$$E_\varepsilon(u) = \int_{\Omega_\varepsilon} |\nabla u|^2, \text{ if } \Sigma \text{ is not singular.}$$

Consider the minimization problem $\mathscr{J}_\varepsilon = \inf\{E_\varepsilon(u), u \in Y^0\}$

Thank to the result of Gromov-Schoen (if $\Sigma = \Sigma_\infty$), it is achieved, i.e. there exists a map

$$\mathfrak{u}_\varepsilon \in Y_0, \text{ which is a minimizer for } \mathscr{J}_\varepsilon.$$



Proposition

There exists a map $u_* : \Omega_* \to \Sigma$ and a subsequence $\varepsilon_m \to 0$ as $m \to \infty$, such that for every compact set $K \subset \Omega_*$, we have, as $m \to \infty$

$$u_{\varepsilon_m} \to u_* \text{ in strongly in } H^1(K) \text{ and } C^{(0,\alpha)}(K) \text{ for any } 0 < \alpha < 1.$$

The map u_* is locally minimizing, i.e. for every ball $B(z_0, R) \subset \Omega_*$, u_* minimizes the Dirichlet energy among all maps in $H^1(B(z_0, R))$ with the same value on the boundary $\partial B(z_0, R)$. It follows

$$\frac{\partial}{\bar{\partial} z}\omega(u_*) = 0 \text{ on } \Omega_*.$$

The Hopf differential $\omega(\mathfrak{u}_*)$

The map $\omega(\mathfrak{u}_*)$ is a central tool for the analysis of properties of $\mathfrak{u}_*$. Since $\omega(\mathfrak{u}_*)$ is meromorphic on $\mathbb{R}^2$, it will be completely determined by its behavior near its poles $\{a_i\}_{i=1}^{\ell}$ and its behavior at infinity.

Proposition

There exists ℓ numbers $(\theta_i(a_1,\dots,a_\ell))_{i=1}^{\ell}$ such that the function $\omega(\mathfrak{u}_*)$ writes

$$\omega(\mathfrak{u}_*)(z) = -\sum_{i=1}^{\ell}\left(\frac{4}{(z-a_i)^2} + \frac{\theta_i(a_1,\dots,a_\ell)}{z-a_i}\right).$$

Moreover, we have

$$\omega(\mathfrak{u}_*)(z) \sim -\frac{|\gamma_\infty|^2}{z^2} \text{ as } |z| \to \infty.$$

Main problem: Determine the "functions" $\theta_i(a_1,\dots,a_\ell)$.

Two additional relations

More precisely, the problem is to determine, among all second order differentials of the form

$$\omega(\mathfrak{u}_*)(z) = -\sum_{i=1}^{\ell}\left(\frac{4}{(z-a_i)^2} + \frac{\theta_i}{z-a_i}\right), \theta_1, \dots, \theta_\ell \in \mathbb{C},$$

the ones which are the Hopf differential of a map $\mathfrak{u}_*$.

Matching the expansions as $z \to \infty$ of termes of order -1 and -2 in z, we are led to two additional relations

$$\begin{cases} \sum_{i=1}^{n^2} \theta_i(a_1, \dots, a_{n^2}) = 0 \\ \text{and} \\ \sum_{i=1}^{n^2} \theta_i(a_1, \dots, a_{n^2}) a_i = |\gamma_\infty| - \ell. \end{cases}$$

Hence, we should have only $\ell - 2$ free parameters.

If $\Sigma = \mathbb{S}^1$, the computation of B-Brezis-Hélein (94) show that $\omega(\mathfrak{u}_*)$ is the square of meromprphic function actually

$$\begin{cases} \omega(\mathfrak{u}_*) = \Lambda(\mathfrak{u}_*)^2, \text{ with} \\ \Lambda(\mathfrak{u}_*) = -i \sum_{i=1}^{\ell} \frac{1}{z - a_i}. \end{cases}$$

Expanding, we obtain

$$-\pi\omega(u_*) = \sum_{i=1}^{\ell} \frac{1}{(z-a_i)^2} + \sum_{i=1}^{\ell} \frac{\theta_i(a_i, \ldots, a_\ell)}{z - a_i},$$

where

$$\theta_i(a_i, \ldots, a_\ell) = \sum_{j \neq i} \frac{2}{a_i - a_j}.$$

The function θ_i is hence, for $\Sigma = \mathbb{S}^1$, a uniquely defined meromorphic (actually rational) function of the points $a_1, \ldots, a_\ell$.

For $\Sigma = \Sigma_\infty$ (and singular one-dimensional targets in general) Mehdi Trense was able to prove in his thesis :

Theorem (Trense, 2025)

The coefficients θ_i are (possibly multi-valued) meromorphic functions of the points $a_1, \dots, a_\ell$.

In general, $\omega(\mathfrak{u}_*)$ is NOT the square of a meromorphic function... and therefore one has to take a square root(using suitable Riemann surfaces). Recall, indeed that, if $u(x) \neq 0$, then we may write

$$\Lambda(u) = \pm \left[\frac{\partial u}{\partial x_1} - i \frac{\partial u}{\partial x_2} \right] \cdot \vec{e}_{u(x)},$$

But the sign is not clearly determined **globally**.

The arguments relies in particular of result of Strebel and Jenkins.

Next a few remarks leading to the proof.

we may write, with $c = |\gamma_\infty|$,

$$\omega(\mathfrak{u}_\star) = \mathbf{W}_{\theta,\mathbf{a}} \text{ with } \mathbf{W}_{\theta,\mathbf{a}}(z) = \mathbf{c}^2 \frac{\Pi_{\theta,\mathbf{a}}(z)}{\mathbf{P}^2_{\mathbf{a}}(z)}, \forall z \in \Omega_\star(\mathbf{a}), \tag{9}$$

where we have set $\mathbf{P}_{\mathbf{a}}(z) = \prod_{i=1}^{\ell}(z - a_i)$ and

$$\begin{aligned} \mathbf{c}^2\Pi_{\theta,\mathbf{a}}(z) &= \sum_{i=1}^{\ell}\left(-4\mathbf{Q}^2_{i,\mathbf{a}}(z) + \theta_i \mathbf{Q}_{i,\mathbf{a}}(z)\mathbf{P}_{\mathbf{a}}(z)\right), \text{ with} \\ \mathbf{Q}_{i,\mathbf{a}}(z) &= \prod_{j\neq i}(z - a_j) = \frac{\mathbf{P}_{\mathbf{a}}(z)}{z - a_i}. \end{aligned} \tag{10}$$

We notice that $\deg(\mathbf{P}_{\mathbf{a}}) = \ell$, $\deg(\mathbf{Q}_{i,\mathbf{a}}) = \ell - 1$ and hence $\deg(\Pi_{\theta,\mathbf{a}}) = 2\ell - 2$. The coefficient of order $z^{2\ell-2}$ of $\Pi_{\theta,\mathbf{a}}$ is equal to 1

$\Longrightarrow$

$\Pi_{\theta,\mathbf{a}}$ has $2\ell - 2$ roots $\sigma_1, \dots, \sigma_{2\ell-2}$, which are the zeroes of $\omega(\mathfrak{u}_*)$ and

$$\Pi_{\theta,\mathbf{a}}(z) = \prod_{k=1}^{2\ell-1}(z - \sigma_k).$$

Another parametrization of $\Pi_{\theta,\mathbf{a}}$

By Lagrange interpolation, there exists an auxiliary variable

$$\zeta = (\zeta_0, \zeta_1, \cdots, \zeta_{\ell-3}) \in \mathbb{C}^{\ell-2}, \text{ such that}$$

$$\mathbf{c}^2 \Pi_{\theta,\mathbf{a}}(z) = \mathbf{H_a}(z,\zeta), \text{ for } z \in \mathbb{C}, \tag{11}$$

where $\mathbf{H_a} \in \mathbb{C}[z,\zeta]$ is defined by

$$\begin{cases} \mathbf{H_a}(z,\zeta) = -\underbrace{\sum_{i=1}^{\ell} \left(4\mathbf{Q}_{i,\mathbf{a}}(a_i)\right) \mathbf{Q}_{i,\mathbf{a}}(z)}_{\text{independant of } \zeta} + \mathbf{Z}_\zeta(z) \mathbf{P_a}(z), \text{ with} \\ \mathbf{Z}_\zeta(z) = \mathbf{c}^2 z^{\ell-2} + \sum_{j=0}^{\ell-3} \zeta_j z^j, \forall z \in \mathbb{C}. \end{cases} \tag{12}$$

The problem becomes : Find all values $\zeta_\star$ of ζ which corresponds to a map $\mathfrak{u}_*$.

If $\ell = 2$, (11) still holds with $\mathbf{H_a}$ with $\mathbf{Z} = -\mathbf{c}$.

We have hence

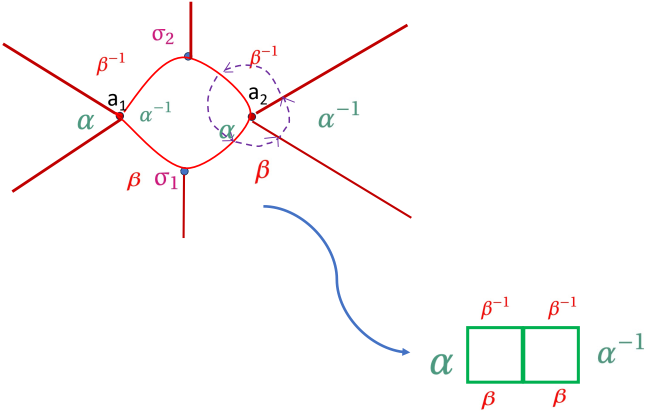
$$\theta_i = \mathbf{L}_i \left(\sum_{j \neq i} \frac{1}{a_i - a_j} \right) + \frac{\mathbf{Z}_\zeta(a_i)}{\mathbf{Q}_i(a_i)}, \text{ for } i = 1, \ldots, \ell. \tag{13}$$

In particular, θ_i is a **rational function** of **a** and ζ.

The case of two singularities $\ell = 2$

In the case $\ell = 2$, we derive

$$\theta_1 = \frac{2}{a_2 - a_1}, \theta_2 = -\frac{2}{a_2 - a_1}.$$



We retrieve the map u_*, by integration, introducing the function

$$F(z) = \int_{z_0}^{z} \lambda(z)dz, \text{ where } z_0 \text{ is arbitrary in } \Omega_*. \tag{14}$$

Since λ is locally holomorphic, we have

$$\frac{\partial F}{\partial z} = \lambda, \tag{15}$$

Locally, if u_* is far from 0, the image of u can arc-length parametrized by φ:

$$\begin{aligned} F(z) &= \int_{z_0}^{z} ((\varphi \circ u)_{x_1} - i(\varphi \circ u)_{x_2})(dx_1 + idx_2) \\ &= \Big[\int_{z_0}^{z} ((\varphi \circ u)_{x_1} dx_1 + (\varphi \circ u)_{x_2} dx_2) + i \int_{z_0}^{z} ((\varphi \circ u)_{x_1} dx_2 - (\varphi \circ u)_{x_2} dx_1) \Big]. \end{aligned} \tag{16}$$

It we take the real part, we obtain

$$\Re(F(z)) = \int_{z_0}^{z} ((\varphi \circ u)_{x_1} dx_1 + (\varphi \circ u)_{x_2} dx_2) = (\varphi \circ u)(z) - (\varphi \circ u)(z_0). \tag{17}$$

Hence, the local arc-length of u is determined by $\Re(F(z))$.

Suppose that at $x=0$, $\omega(\mathfrak{u}_*)$ as a zero of multiplicity 1, hence order 1. Using a holomorphic change of coordinates, we write

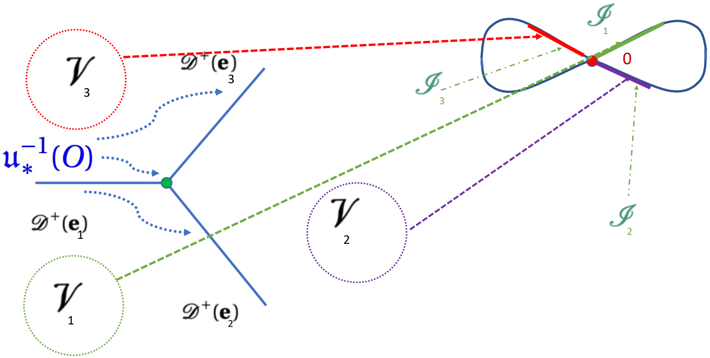
$$\omega(z)=z \text{ near } 0, \text{ so that } F(z)=z^{\frac{3}{2}}.$$

The solutions to the equation

$$\Re(F(z))=0 \tag{18}$$

are the three half lines $D_1^j=\{z=-r, \text{ for } r\geq 0\}$, $D_2^j=\{z=r\exp\frac{i\pi}{3},\ r\geq 0\}$ and $D_3^j=\{z=r\exp-\frac{i\pi}{3},\ r\geq 0\}$. that is, for some neighborhood $\mathscr{U}_j$ of σ_j, we have

$$\mathfrak{u}_*^{-1}(0)\cap\mathscr{U}_j=(D_1^j\cup D_2^j\cup D_3^j)\cap\mathscr{U}_i. \tag{19}$$


$\mathcal{V}_3$
$\mathscr{D}^+(\mathbf{e}_3)$
$u_*^{-1}(O)$
$\mathscr{D}^+(\mathbf{e}_1)$
$\mathcal{V}_1$
$\mathscr{D}^+(\mathbf{e}_2)$
$\mathcal{V}_2$
$\mathscr{I}_1$
$\mathscr{I}_3$
0
$\mathscr{I}_2$

The best way to work with the square root $\lambda(u_*)$ is to introduce the Riemann surface

$$\mathfrak{S} = \left\{(y,z) \in \mathbb{C}\times\mathbb{C}, \text{s.t. } y^2 = \Pi_{\theta,\mathbf{a}}(z) = \prod_{k=1}^{2\ell-2}(z-\sigma_k)\right\}, \ \sigma_1,\dots,\sigma_{2\ell-2} \text{ zeroes of } u_*.$$

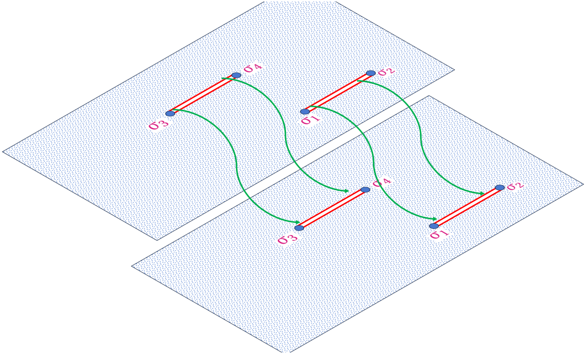


Figure: $\ell = 3$, *4 roots*

For $k_1 \neq k_2$, consider two roots σ_i, σ_j, and

$$A_{k_1,k_2} = \int_{\sigma_{k_1}}^{\sigma_{k_2}} F(z)\mathrm{d}z = \int_{\sigma_{k_1}}^{\sigma_{k_2}} \frac{\sqrt{\Pi_{\theta,\mathbf{a}}(z)}}{\mathbf{P_a}(z)}\mathrm{d}z$$
$$= \mathbf{c}^{-1} \int_{\sigma_{k_1}(\zeta_\star)}^{\sigma_{k_2}(\zeta_\star)} \frac{\sqrt{\mathbf{H}_{\zeta_\star,\mathbf{a}}(z)}}{\mathbf{P_a}(z)}\mathrm{d}z \equiv A_{k_1,k_2}(\zeta_\star). \quad (20)$$

Since $\mathfrak{u}_*(\sigma_{k_1}(\zeta_\star) = \mathfrak{u}_*(\sigma_{k_2}(\zeta_\star)) = O$, this implies

$$\Re(A_{k_1,k_2})(\zeta_\star) \in \mathbb{Z}^*.$$

Mehdi Trense was able to prove that the imaginary part vanishes, so that, this yields

$$A_{k_1,k_2}(\zeta_\star) \in \mathbb{Z}^*.$$

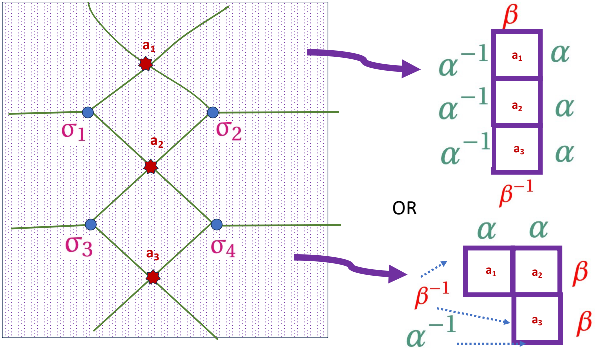
a condition which characterizes $\zeta_\star$ and hence $\theta_i(a_1,\ldots,a_\ell)$.

When $\ell = 3$, the number of roots σ is $2\ell - 2 = 4$,

$$\mathbf{H_a}(z, \zeta) = \mathbf{c}^2 z + \zeta.$$

and the genus g of $\mathfrak{S}$ is

$$g = 1.$$



Thank you for your attention!

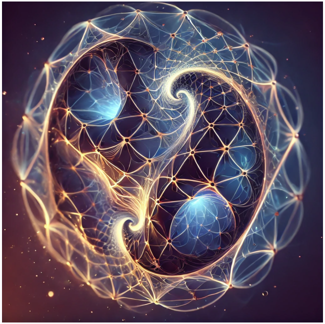


Figure: *An "artistic" view of an hyperelliptic Riemann surface according to chatGPT*