Atiyah classes of homotopy modules

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Outline

- **Introduction:** Words in the title
- **Differential Graded Geometry:** A concise description of several different objects
- **Atiyah classes:** Obstructions to the existence of certain structures in Geometry and Lie Theory
- **Representations up to homotopy:** Higher Curvatures and examples

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Evolution of the class

- **Origin:** Atiyah (1957) constructed the Atiyah class to characterize the obstrcution to the existence of holomorphic connections on a holomorphic vector bundle.
- **Motivation for studying Atiyah classes**
  - Kontsevich (1997): relation between Atiyah class in complex geometry and Duflo Isomorphism in Lie theory (see the book by Calaque-Rossi)
  - Kapranov (1997): Atiyah class of a Kähler manifold induces an $L_\infty-$ structure, important for Rozansky-Witten invariants
- **Beyond complex geometry:**
  - Chen-Stiénon-Xu (2012): extended the theory of Atiyah classes from complex geometry to Lie (algebroid) pairs
  - Mehta-Stiénon-Xu (2015): extended the theory of Atiyah classes from complex geometry to dg Lie algebroids

Q&A

Question What is the Atiyah class of representations up to homotopy and its relation to the Atiyah class of dg Lie algebroids?

Answers

- Liao-Stiénon-Xu (2016), Batakidis-Voglaire (2016):
  The Atiyah class of the associated Fedosov dg Lie algebroid can be identified with the Atiyah class of a given Lie pair
- Liao (2022):
  The Atiyah class of the associated pullback dg Lie algebroid can be identified with the Atiyah class of a given Lie pair
- Batakidis-Lavau (2024):
  Definition of the Atiyah class for arbitrary representation up to homotopy and a result that extends the previous answers.

Lie pair (L, A) = Lie algebroid $L \to M$ + Lie subalgebroid $A \to M$.

1. Given a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, $(\mathfrak{g}, \mathfrak{h})$ is a Lie pair over $M = \{pt\}$.
2. If F is a regular foliation on M, (TM, TF) is a Lie pair over M.
3. If $\mathcal{F}_1, \mathcal{F}_2$ are transversal foliations on M, then $T\mathcal{F}_1 \bowtie T\mathcal{F}_2 \simeq TM$ is a matched pair.
4. If (P, π) is a Poisson $G-$ space (Poisson P w/ a Poisson $G-$ action), then $A = (T^*P)_\pi$ $B = P \times \mathfrak{g}$ is a matched pair.
5. If X is complex manifold then $(TX \otimes \mathbb{C}, T^{0,1}X)$ is a Lie pair and $A = T^{0,1}X, B = T^{1,0}X$ is a matched pair.

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Representations

→ Lie algebroids come with a cohomology theory (recovering Lie algebra, de Rham, Dolbeault cohomologies)

Definition

A connection of a Lie algebroid L on a vector bundle E is a linear map $\nabla : \Gamma(L) \otimes \Gamma(E) \to \Gamma(E)$ such that

$$\nabla_{fl} e = f \nabla_l e, \quad \nabla_l fe = f \nabla_l e + \rho(l)(f)e.$$

When the curvature $R^{\nabla} : L \wedge L \to \mathrm{End}(E)$ vanishes, this is a Lie algebroid representation on E.

Example (Bott connection): For a Lie pair (L, A), take $E = L/A$, $\nabla_a^{Bott} \bar{l} = \overline{[a, l]}$. It is a flat $A-$ connection.
→ The section $R_{1,1}^{\nabla} \in \Gamma(A^* \otimes (L/A)^* \otimes \mathrm{End}(E))$ is a 1-cocycle whose induced cohomology (a.k.a. Atiyah) class is independent of the choice of $L-$ connection extending ∇^A.

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Definition

A *representation up to homotopy of A* – or *homotopy A-module* – is a (split, finite dimensional) graded vector bundle E with vector bundle morphism/chain map $\partial : E_\bullet \to E_{\bullet+1}$, together with an A-connection ∇ on E, and a family of forms $\left(\omega_A^{(k)}\right)_{k\geq 2} \in \Omega^k(A, \mathrm{End}(E))_1$ of total degree $+1$ such that

$$[d_A^\nabla, \partial] = 0 \tag{1}$$

$$R_A^{(k)} + \left[\partial, \omega_A^{(k)}\right] = 0 \qquad \text{for every } k \geq 2 \tag{2}$$

$R_A^{(k)}$ is the *curvature k-form* associated to the connection $k-1$-form $\omega_A^{(k-1)}$:

$$R_A^{(k)} = d_A \omega_A^{(k-1)} + \sum_{\substack{1\leq s,t\leq k-1 \\ s+t=k}} \omega_A^{(s)} \wedge \omega_A^{(t)}.$$

Let M be a smooth manifold with structure sheaf $\mathcal{O}_M$. Let $\hat{S}(V^*)$ denote the algebra of formal power series on some fixed $\mathbb{Z}-$ graded vector space V.

Definition

A $\mathbb{Z}-$ graded manifold $\mathcal{M}$ with *body* M is a sheaf $\mathcal{R}$ of $\mathbb{Z}-$ graded commutative $\mathcal{O}_M-$ algebras over M such that $\mathcal{R}(U) \cong \mathcal{O}_M(U)\hat{\otimes}\hat{S}(V^*)$ for all sufficiently small open subsets $U \subset M$.

Definition

A dg manifold is a $\mathbb{Z}-$ graded manifold $\mathcal{M}$ endowed with a vector field $Q \in \mathfrak{X}(\mathcal{M})$ of degree $+1$ such that $[Q, Q] = 2Q \circ Q = 0$.

We say that Q is a *homological* vector field on the graded manifold $\mathcal{M}$.

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DG Geometry - Examples of dg manifolds

- If L is a Lie algebroid, then $\mathcal{M} = L[1]$ is a dg manifold.
  ✓ its algebra of functions: $C^\infty(L[1]) \cong \Lambda^\bullet Ł^*$
  ✓ its homological vector field: $Q = \mathrm{d}_L$ (Lie al/oid cohomology)
- If M is a smooth manifold, then $\mathcal{M} = T_M[1]$ is a dg manifold.
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- If X is a complex manifold, then $\mathcal{M} = T_X^{0,1}[1]$ is a dg manifold.
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- If $s \in \Gamma(E)$, then $E[-1]$ is a dg manifold (the derived zero locus of s)
  ✓ its algebra of functions $C^\infty(E[-1])$ with homological vector field $Q = i_s$ is
  $\cdots \xrightarrow{i_s} \Lambda^2 E^* \xrightarrow{i_s} \Lambda^1 E^* \xrightarrow{i_s} C^\infty(M)$

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Connections on graded manifolds

Since Lie pairs, regular foliations, complex manifolds, Courant algebroids and etc are included in the category of graded manifolds...

Theorem (B. - Voglaire)

For every Lie pair (L, A), $\mathcal{M} = L[1] \oplus L/A$ is a dg manifold s.t. $\iota : A[1] \to \mathcal{M}$ and $p : \mathcal{M} \to L[1]$ are morphisms of dg manifolds

Definition

A connection on a graded manifold $\mathcal{M}$ is a linear map $\nabla : \mathfrak{X}(\mathcal{M}) \otimes \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ **of degree 0** such that

$$\nabla_{fX} Y = f \nabla_X Y, \quad \nabla_X fY = X(f)Y + (-1)^{|X||f|} f \nabla_X Y$$

for all homogeneous $f \in C^\infty(\mathcal{M}), X, Y \in \mathfrak{X}(\mathcal{M})$.

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✓ Choose a torsion-free connection on $\mathcal{M}$ (recall that $\Gamma(T\mathcal{M}) := \mathfrak{X}(\mathcal{M})$)

$$\nabla : \ \mathfrak{X}(\mathcal{M}) \otimes \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M}).$$

✓ Consider the section $\mathrm{At}^{\nabla}_{(\mathcal{M},Q)}$ of $\mathrm{Hom}(S^2(T\mathcal{M}, T\mathcal{M})$ of degree $+1$ defined by

$$\mathrm{At}^{\nabla}_{(\mathcal{M},Q)}(X,Y) = \mathcal{L}_Q(\nabla_X Y) - \nabla_{\mathcal{L}_Q X} Y - (-1)^{|X|}\nabla_X(\mathcal{L}_Q Y)$$

for all homogeneous $X, Y \in \mathfrak{X}(\mathcal{M})$.

✓ Since $\mathcal{L}_Q \circ \mathcal{L}_Q = 0$, $\mathrm{At}^{\nabla}_{(\mathcal{M},Q)} = \mathcal{L}_Q \nabla$ is a 1-cocycle of the cochain complex

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✓ Its cohomology class in independent of the choice of ∇.

Atiyah classes in dg Geometry

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A connection ∇ on a dg manifold $(\mathcal{M}, Q)$ is said to be compatible with Q if

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Definition

The Atiyah class of the dg manifold $(\mathcal{M}, Q)$

$$\alpha_{(\mathcal{M},Q)} := [\mathrm{At}^{\nabla}_{(\mathcal{M},Q)}] \in H^1\left(\Gamma(\mathrm{Hom}(S^2(T\mathcal{M}), T\mathcal{M})), \mathcal{L}_Q\right)$$

is the obstruction to existence of a connection on $\mathcal{M}$ compatible with the homological vector field Q.

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Theorem (B. - Voglaire)

For every Lie pair (L, A) coming from a matched pair and its corresponding dg manifold $L[1] \oplus L/A$, the Atiyah classes "coincide".

Theorem (Liao - Stiénon - Xu)

For every Lie pair (L, A), there exists an L_∞ quasi-isomorphism

$$tot(\Gamma(\Lambda^\bullet A^*) \otimes \mathcal{T}^\bullet_{poly}) \to tot(\Gamma(\Lambda^\bullet A^*) \otimes \mathcal{D}^\bullet_{poly})$$

whose first Taylor coefficient is $hkr \circ (td^{\nabla}_{L/A})^{\frac{1}{2}}$.

$\rightarrow$ There is a dg manifold version by Stiénon-Xu.

Atiyah classes

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For every Lie pair (L, A) coming from a matched pair and its corresponding dg manifold $L[1] \oplus L/A$, the Atiyah classes "coincide".

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For every Lie pair (L, A), there exists an L_∞ quasi-isomorphism

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Atiyah classes - Main results

Theorem (B. - Lavau)

Let (L, A) be a Lie pair, (E, D_A) be a representation up to homotopy of A, and D_L be a superconnection of L on E extending D_A. Then, D_L induces a total degree $+1$ differential s on

$$\widehat{\Omega}(E) = \bigoplus_{j=-\infty}^{\infty} \bigoplus_{k=0}^{\mathrm{rk}(A)} \Omega^k(A, A^\circ \otimes \mathrm{End}_j(E))$$

making $A^\circ \otimes \mathrm{End}(E)$ a r.u.t.h. of A.
$\exists$ a 1-cocycle $\alpha = \sum_k \alpha^{(k)}$ of $\left(\widehat{\Omega}(E), s\right)$ such that:

1. $[\alpha] \in H^1\left(\widehat{\Omega}(E), s\right)$ *does not depend on the choice of* D_L
2. $[\alpha] = 0$ *iff* $\exists$ *a homotopy A-compatible L-superconnection*
3. *If E is "regular" and a resolution of $H^0(E, \partial)$, then*

$$H^p\left(\widehat{\Omega}(E), s\right) \simeq H^p\left(A, A^\circ \otimes \mathrm{End}(K)\right), \quad [\alpha] \mapsto [\mathrm{at}_K]$$

Contractions

A contraction is the data

- two cochain complexes (V, d_V), (W, d_W)
- two chain maps $\tau : V \hookrightarrow W$, $\sigma : W \twoheadrightarrow V$
- a homotopy operator $h : W \to W$ of degree -1 s.t.

$$\sigma\tau = id_V, \quad id_W - \tau\sigma = hd_W + d_W h, \quad \sigma h = 0, \quad h\tau = 0, h^2 = 0.$$

Tensor trick: Out of this you can build contractions of tensors on V, W.

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Theorem (B. - Lavau)

With the same conditions (E is a resolution of $H^0(E, \partial)$) set $\mathcal{E} = \pi^* E$ (resp. $\mathcal{K} = \pi^* K$) to be the pullback of E (resp. K) over $\pi : A[1] \to M$.

1. *The operator $d = D_A - \partial$ is a small perturbation of a contraction between $(\Gamma_{A[1]}(\mathcal{K}), 0)$ and $(\Gamma_{A[1]}(\mathcal{E}), \partial)$. The perturbed contraction forms a contraction data over $(\mathcal{C}^\infty(A[1]), d_A)$.*
2. *There is a quasi-isomorphism*

$$\left(\widehat{\Omega}(E)^{\bullet,\bullet}, s\right) \simeq \left(\Gamma_{A[1]}\left(\mathcal{L}^* \otimes \mathrm{End}(\mathcal{E})\right), \mathcal{Q} + [D_A, .]\right).$$

   where $\mathcal{Q}$ is the homological vector field associated to the dg Lie algebroid $\mathcal{L} \to A[1]$.

Thank you all!