Some results on the L_p-Brunn-Minkowski inequality for intrinsic volumes and the L_p-Christoffel-Minkowski problem

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Basic convexity

- Support function of a convex body K:

$$h_K(x) = \max_{y \in K} \langle x, y \rangle, \qquad x \in \mathbb{R}^n$$

- Wulff shape of $f : \mathbb{S}^{n-1} \to (0, \infty)$:

$$W(f) := \{x \in \mathbb{R}^n : \langle x, u \rangle \leq f(u), \forall u \in \mathbb{S}^{n-1}\}.$$

- Let $\lambda \in (0,1)$, $p \in \mathbb{R}$. L_p-sum ($p = 1$ is the classical Minkowski sum):

$$(1-\lambda) \cdot K +_p \lambda \cdot L = W\big(((1-\lambda)h_K^p + \lambda h_L^p)\big)^{1/p}\big).$$

- L_0-sum: $(1-\lambda) \cdot K +_0 \lambda \cdot L = W(h_K^{1-\lambda} h_L^{\lambda})$
- In the case $p \geq 1$, $((1-\lambda)h_K^p + \lambda h_L^p))^{1/p}$ is convex, hence a support function. Much more difficult to compute the support function of the L_p-sum, if $p < 1$.

L_p-Brunn-Minkowski inequality

- L_p-Brunn-Minkwoski inequality for $p \geq 1$

$$V((1-\lambda)\cdot K +_p \lambda \cdot L) \geq \left((1-\lambda)V(K)^{p/n} + \lambda V(L)^{p/n}\right)^{n/p},$$

for all convex bodies K, L that contain the origin.

- Follows from the clasical case $p = 1$ with a homogeneity argument and the inclusion (if $q > p$)

$$(1-\lambda)\cdot K +_q \lambda \cdot L \supseteq (1-\lambda)\cdot K +_p \lambda \cdot L.$$

- False for $p < 1$ but conjectured to hold under symmetry (Böröczky, Lutwak, Yang, Zhang),
- ...interpeted as

$$V((1-\lambda)\cdot K +_0 \lambda \cdot L) \geq V(K)^{1-\lambda}V(L)^{\lambda}$$

when $p = 0$.

L_p-Brunn-Minkowski inequality

- False for $p < 0$ even under symmetry.
- If the inequality holds for p then it holds for any $q > p$.
- Holds in the plane even for $p = 0$ (Böröczky-Lutwak-Yang-Zhang)
- Holds for unconditional convex bodies even for $p = 0$ (Saroglou)
- Holds for p very close to 1 (Kolesnikov-Milman; Chen-Huang-Li-Liu)

Principal radii of curvature and area measures

- For $f \in C^2(\mathbb{R}^n)$ positively homogeneous and $p \in \mathbb{S}^{n-1}$ consider an orthonormal basis $\{e_1, \dots, e_{n-1}\}$ of $T_p\mathbb{S}^{n-1}$ and the matrix $\mathcal{A}f(p) = (D_{e_i} D_{e_j} f(p))_{i,k=1}^{n-1} \geq 0$.
- Denote $\lambda_1, \dots, \lambda_{n-1}$ its eigenvalues (called principal radii of curvature if of K if $f = h_K$).
- For $j \in \{1, \dots, n-1\}$ write the renormalized j-th elementary symmetric function of the eigenvalues:

$$s_j(f, p) = \binom{n-1}{j}^{-1} \sum_{1 \leq i_1 < \cdots < i_j \leq n-1} \lambda_{i_1} \dots \lambda_{i_j}.$$

- If $h_K \in C^2(\mathbb{R}^n)$ and $\mathcal{A}f(p) > 0$ for every p then define the j-th area measure of K on $\mathbb{S}^{n-1}$ denoted $S_j(K, \cdot)$ with density $s_j(h_K, \cdot)$.

Intrinsic volumes

- For $j \in \{0, \dots, n\}$ the j-th intrinsic volume of a convex body K is defined by its Steiner polynomial

$$V(K + rB_2^n) = \sum_{j=0}^{n} r^{n-j} \kappa_{n-j} V_j(K),$$

  where κ_d the volume of the d-dimensional euclidean unit ball.
- $V_n(K) = V(K)$
- For $j \in \{1, \dots, n-1\}$ calculated explicitly as

$$V_j(K) = \binom{n}{j} \frac{1}{n\kappa_{n-j}} \int_{\mathbb{S}^{n-1}} h_K \, dS_{j-1}(K, \cdot),$$

L_p-Brunn-Minkowski inequality for intrinsic volumes

- L_p-Brunn-Minkowski inequality for intrinsic volumes:

$$V_j((1-\lambda)\cdot K +_p \lambda \cdot L) \geq \left((1-\lambda)V_j(K)^{p/j} + \lambda V_j(L)^{p/j}\right)^{j/p},$$

where $j \in \{2, \ldots, n-1\}$, interpreted as

$$V_j((1-\lambda)\cdot K +_0 \lambda \cdot L) \geq V_j(K)^{1-\lambda} V_j(L)^{\lambda}.$$

if $p = 0$.

- For $j = 1$, the exact reverse inequality holds (Minkowski's triangle inequality).
- Holds for $p = 1$, hence, as in the case of volume, for $p \geq 1$ (the smaller the p, the stronger the inequality).
- What happens for $p < 1$?
- Bianchini, Colesanti, Pagnini, Roncoroni studied the problem for $0 \leq p < 1$.

L_p-Brunn-Minkowski inequality for intrinsic volumes

They proved the following

- Theorem A (Bianchini, Colesanti, Pagnini, Roncoroni) For $j \in \{2, \dots, n-1\}$ there exists a constant η that depends only on n, such that for all $\lambda \in (0,1)$ and for all symmetric convex bodies L with $\|h_L - 1\|_{C^2} < \eta$, it holds

$$V_j((1-\lambda) \cdot B_2^n +_0 \lambda \cdot L) \geq V_j(B_2^n)^{1-\lambda} V_j(L)^{\lambda}.$$

  This corresponds to $p = 0$. Thus, the same holds for any $0 < p < 1$.

- Theorem B (BCPR) For every $j \in \{2, ..., n-1\}$, there exists $\bar{p} \in (0,1)$ such that for every $p \in (0, \bar{p})$ the L_p-Brunn–Minkowski inequality for intrinsic volumes of symmetric convex bodies does not hold. That is, there exist K, L such that

$$V_j(\frac{1}{2} \cdot K +_p \frac{1}{2} \cdot L) < \left(\frac{1}{2} V_j(K)^{p/j} + \frac{1}{2} V_j(L)^{p/j} \right)^{j/p}$$

Positive results

We improve Theorem A with the following stability result. The one of the two bodies is perturbed (no more a ball) but the inequality remains valid. The range of p is also extended up to the negative value $\frac{2n-j-nj}{n-1}$.

Theorem (1)

(P.-Saroglou) For every $n \geq 3$, every $j \in \{2, 3, \dots, n-1\}$ and every $p \in (\frac{2n-j-nj}{n-1}, 1)$ (notice that $\frac{2n-j-nj}{n-1} < 0$) there is $\delta > 0$ that depends only on n, p, j, such that, for every symmetric convex bodies K, L satisfying $\|h_K - 1\|_{C^2} < \delta$ and $\|h_L - 1\|_{C^2} < \delta$ and every $\lambda \in (0,1)$, it holds

$$V_j((1-\lambda) \cdot K +_p \lambda \cdot L) \geq \left((1-\lambda) V_j(K)^{p/j} + \lambda V_j(L)^{p/j}\right)^{j/p}. \quad (1)$$

Negative results

We also improve Theorem B by proving that L_p-Brunn-Minkowski does not hold for any λ even for p near 1.

Theorem (2)

(P.-Saroglou) For every $n \geq 3$, every $j \in \{2, 3, \ldots, n-1\}$, every $\lambda \in (0,1)$ and every $p \in [0,1)$, there are symmetric convex bodies K and L, s.t.

$$V_j((1-\lambda) \cdot K +_p \lambda \cdot L) < V_j(K)^{1-\lambda} V_j(L)^{\lambda}$$

$$\leq \left((1-\lambda) V_j(K)^{p/j} + \lambda V_j(L)^{p/j}\right)^{j/p}.$$

L_p-Christoffel-Minkowski problem

- For an absolutely continuous Borel measure on the sphere with density g, the L_p-Christoffel-Minkowski problem asks for existence and uniqueness of the equation

$$h_K^{1-p} s_j(h_K, \cdot) = g.$$

- The case $j = n-1$ is the (essentially solved) L_p-Minkowski problem.
- For $j < n-1$, existence is a challenging open problem, while uniqueness would follow from L_p-Brunn-Minkowski inequality.

Theorem (3)

(P.-Saroglou) For $\alpha \in (0,1)$, $p \in [0,1)$, $j \in \{1, \dots, n-2\}$, $(p,j) \neq (0,1)$ there exists $\delta = \delta(n,p) > 0$, such that in the class of symmetric C^2 convex bodies, the equation $h_K^{1-p} s_j(h_K, \cdot) = g$ has a unique solution, provided $g \in C^\alpha$ and $\|g - 1\|_{C^\alpha} < \delta$.

- True if $g = 1$ by Chen and Ivaki-Milman. Our Theorem is a stability result of their.

Proof of Theorem 1 (sketch)

- Closely follows Kolesnikov-Milman.
- We have to prove that

$$\lambda \mapsto pV_j\left((1-\lambda)\cdot K +_p \lambda \cdot L\right)$$

  is concave.
- Define the linear operator

$$T_K^j : C^2 \ni z \mapsto z - \frac{D_{n-1}(\mathcal{A}(zh_K), \mathcal{A}h_K[j-2], Id[n-j])}{s_{j-1}(h_K)} \in C^0.$$

- It suffices to prove that

$$\inf_{z\text{even},\ z\perp 1,\ \|z\|=1} \langle z, T_K^j z\rangle_{L^2} > \frac{j-p}{j-1},$$

w.r.t. $dV_K^j := \frac{1}{n} h_K dS_{j-1}(K)$

Proof of Theorem 1 (sketch)

- T_K^j is symmetric and positive semi-definite with $T_K^j(1) = 0$.
- $-T_{B_2^n}^j = \frac{1}{n-1}\Delta_{\mathbb{S}^{n-1}}$.
- In particular, $-T_{B_2^n}^j$ is uniformly elliptic and the same is true for $-T_K^j$, if K is C^2-close to the ball.
- For such K's, T_K^j has discrete spectrum and the sequence of its eigenvalues tends to infinity.
- The smallest eigenvalue of $T_{B_2^n}^j$ is $\frac{2n}{n-1} > \frac{j-p}{j-1}$ for $p > \frac{2n-j-nj}{n-1}$.
- But the infimum of the Rayleigh quotient is the smallest eigenvalue of T_K^j that corresponds to even functions that are orthogonal to 1.
- Inequality still holds C^2-close to the ball. $\square$

Proof of Theorem 2 (sketch)

The following lemma relies on Minkowski's triangle inequality

Lemma

Let $p \in [0,1]$, $\lambda \in (0,1)$, E be a subspace of $\mathbb{R}^n$, A, C be symmetric convex bodies in E and B, D be symmetric convex bodies in $E^\perp$. Then,

$$(1-\lambda)\cdot(A+B)+_p\lambda\cdot(C+D) = ((1-\lambda)\cdot A+_p\lambda\cdot C)+((1-\lambda)\cdot B+_p\lambda\cdot D),$$

where $(1-\lambda)\cdot A +_p \lambda\cdot C$ and $(1-\lambda)\cdot B +_p \lambda\cdot D$ denote L^p-sum in E and $E^\perp$ respectively.

- We prove Theorem 2 by induction on n. Trivial if $j = 1$, hence if $n = 2$.

Proof of Theorem 2 (sketch)

- In the inductive step fix $j \in \{2, \dots, n-1\}$. and assume that there exist symmetric convex bodies K, L in $\theta^{\perp}$, satisfying

$$V_{j-1}^{\theta^{\perp}}((1-\lambda)\cdot K +_p \lambda \cdot L) < V_{j-1}^{\theta^{\perp}}(K)^{1-\lambda} V_{j-1}^{\theta^{\perp}}(L)^{\lambda}.$$

- By Minkowski-adding to K and L a sufficiently large line segment parallel to θ and applying the Lemma we prove that these "elongated" bodies do not satisfy L_p-Brunn-Minkowski.

Proof of Theorem 3 (sketch)

We use the following proposition which, in our opinion, is of particular interest too:

Theorem

(P.-Saroglou) Let $p \in [0,1)$, $j \in \{1,\dots,n-2\}$ with $(p,j) \neq (0,1)$. If K is a symmetric convex body satisfying $h_K \in C^2$, $\mathcal{A}h_K > 0$ everywhere on $\mathbb{S}^{n-1}$, $h_K^{1-p} s_j(K,\cdot) = g$ for some even measurable g then there exists a constant $C > 0$, that depends only on $n, p, \|\log g\|_{L^\infty}$, such that

$$C^{-1} \leq h_K(x) \leq C, \qquad \forall x \in \mathbb{S}^{n-1}.$$

Proof of Theorem 3 (sketch)

- LOCAL UNIQUENESS: By the L^p-BM inequality for intrinsic volumes close to the ball, there exists $\eta > 0$, such that if h_{K_1}, h_{K_2} are C^2 even support functions that satisfy $\|h_{K_i} - 1\|_{C^2} < \eta$, $i = 1, 2$ and

$$h_{K_1}^{1-p} s_j(h_{K_1}, \cdot) = h_{K_2}^{1-p} s_j(h_{K_2}, \cdot),$$

  then $K_1 = K_2$.
- EXISTENCE: For g even with $\|g - 1\|_{C^\alpha} < \delta_{n,p,\alpha}$, existence of $C^{2,\alpha}$ even solution in $h_K^{1-p} s_j(h_K, \cdot) = g$ follows applying the inverse function theorem for Banach spaces on

$$\Xi : C^{2,\alpha} \ni h \mapsto s_j(h, \cdot) h^{1-p} \in C^\alpha$$

  after observing that its Fréchet derivative at $h = 1$ is

$$D\Xi|_1(w) = \frac{j}{n-1} \Delta_{\mathbb{S}^{n-1}} w + (j - p + 1) w$$

  and is a bounded linear isomorphism.

Proof of Theorem 3 (sketch)

- In particular, in the case $p = 1$ (that is, $\|s_j(h_K, \cdot) - 1\|_{C^\alpha} < \delta_{n,\alpha}$) we easily obtain $\|h_K\|_{C^{2,\alpha}} \leq C$
- GLOBAL UNIQUENESS: Assume that Theorem 3 is not true. Then, there exists a sequence $\{h_{K_m}\}$ of C^2 even support functions that satisfy $h_{K_m}^{1-p} s_j(h_{K_m}, \cdot) = g_m$, $g_m \to 1$ in C^α norm and the previous equation has multiple solutions.
- By local uniqueness, we may assume that $\|h_{K_m} - 1\|_{C^2} > \eta$, for all m.
- We have $\|h_{K_m}\|_{L^\infty} < C$, thus by taking subsequence we may assume that $h_{K_m} \to h_K$ in Hausdorff, for some symmetric convex body K (Blashke selection theorem).

Proof of Theorem 3 (sketch)

- But $\nabla h_K = 0$ a.e. on $\mathbb{S}^{n-1}$ (by Ivaki-Milman), hence $K = B_2^n$.
- Now h_{K_m} are uniformly bounded support functions converging in C^0 to 1, hence they also converge in C^α to 1 (easy).
- It follows that $s_j(h_{K_m}, \cdot) = g_m h_{K_m}^{1-p}$ converge to 1 in C^α, hence $\|h_{K_m}\|_{C^{2,\alpha}} \leq C$ and thus there is a subsequence $h_{K_{r_m}} \xrightarrow{C^2} 1$, a contradiction.

Thank you!!!!!!

Thank you for your attention!!!!!!