Energy spectrum and quantitative estimates for harmonic maps

Melanie Rupflin, University of Oxford

16th Panhellenic Geometry Conference

Basic questions for variational problems:

Given an energy E, what can we say about

- minimisers and critical points of E
- the energy spectrum

$$\Xi_E := \{E(u) : u \text{ critical point } \}$$

- the behaviour of
  - minimising sequences
  - almost critical points
  - gradient flow $\partial_t u = -\nabla E(u)$

Are "almost minimisers" almost given by minimisers?

If we know that the minimum of E is achieved, we can ask whether

$$E(u) \approx E_{min} \quad \Rightarrow \quad u \approx \text{ a minimiser ?}$$

Are "almost minimisers" almost given by minimisers?

If we know that the minimum of E is achieved, we can ask whether

$$E(u) \approx E_{min} \quad \Rightarrow \quad u \approx \text{ a minimiser ?}$$

Qualitative version: Do we have

$$E(u_i) \to \min E \Rightarrow dist(u_i, \{minimisers\}) \to 0?$$

Are "almost minimisers" almost given by minimisers?

If we know that the minimum of E is achieved, we can ask whether

$$E(u) \approx E_{min} \quad \Rightarrow \quad u \approx \text{ a minimiser ?}$$

Qualitative version: Do we have

$$E(u_i) \to \min E \Rightarrow dist(u_i, \{minimisers\}) \to 0?$$

Quantitative version: Can we bound

$$dist^2(u, \{minimisers\}) \leq C\delta_u^\alpha$$

for **energy defect** $\delta_u = E(u) - E_{min}$ and some $\alpha > 0$?

Are "almost minimisers" almost given by minimisers?

If we know that the minimum of E is achieved, we can ask whether

$$E(u) \approx E_{min} \quad \Rightarrow \quad u \approx \text{ a minimiser ?}$$

Qualitative version: Do we have

$$E(u_i) \to \min E \Rightarrow dist(u_i, \{minimisers\}) \to 0?$$

Quantitative version: Can we bound

$$dist^2(u, \{minimisers\}) \leq C\delta_u^\alpha$$

for **energy defect** $\delta_u = E(u) - E_{min}$ and some $\alpha > 0$?

In past 20 years: Important results in many fields of geometry and analysis, including Isoperimetric problems, spectral problems, Yamabe problem, umbilical surfaces, Elasticity, Sobolev embeddings ...
(by Cianchi, Fusco, Maggi, Pratelli, Figalli, Novaga Capella, Otto, Müller, Brasco, De Philippis, Conti, Dolzmann, De Lellis, Székelyhidi, Topping, Engelstein, Neumayer, Spolaor, Lamm, Nguyen, Luckhaus, Zemas, ...)

Almost Critical points:

If u "almost solves" the Euler-Lagrange equation, i.e. is s.t. $\nabla E(u) \approx 0$, does this imply that

- u is close to an exact solution, i.e. a critical point
- $E(u)$ is close to a critical value?

Almost Critical points:

If u "almost solves" the Euler-Lagrange equation, i.e. is s.t. $\nabla E(u) \approx 0$, does this imply that

- u is close to an exact solution, i.e. a critical point
- $E(u)$ is close to a critical value?

Quantitative version = Łojasiewicz estimates

Can we bound

$$\begin{aligned} \text{dist}(u, \{\text{cricital points}\}) &\leq C\|\nabla E(u)\|^{\gamma_1} \\ |E(u) - c^*| &\leq C\|\nabla E(u)\|^{\gamma_2} \quad ? \end{aligned}$$

Almost Critical points:

If u "almost solves" the Euler-Lagrange equation, i.e. is s.t. $\nabla E(u) \approx 0$, does this imply that

- u is close to an exact solution, i.e. a critical point
- $E(u)$ is close to a critical value?

Quantitative version = Łojasiewicz estimates

Can we bound

$$\text{dist}(u, \{\text{cricital points}\}) \leq C\|\nabla E(u)\|^{\gamma_1}$$
$$|E(u) - c^*| \leq C\|\nabla E(u)\|^{\gamma_2} \quad ?$$

- Applications: Convergence of gradient flows, properties of energy spectrum, Quantitative rigidity estimates
- For **analytic** energies in **non-singular** situations: Many results known based on approach of L. Simon ('82)
- Few results that apply in singular situations:
  harmonic maps $S^2 \to S^2$ Topping '04, Waldron'23
  MCF: Colding-Minicozzi '15, Chodosh-Schulze '19
  Optimal Sobolev embeddings: Figalli-Glaudo'19, Deng-Sun-Wei '21

Warm up: A simple energy:

Warm up: A simple energy:

Consider $E : \mathbb{R} \to \mathbb{R}$

$$E(u) = cu^2, \qquad \text{for } c > 0.$$

Then

$$\delta_u := E(u) - E_{min} = cu^2, \quad \text{dist} = |u|, \quad \|\nabla E(u)\| = 2c|u|.$$

Warm up: A simple energy:

Consider $E : \mathbb{R} \to \mathbb{R}$

$$E(u) = cu^2, \qquad \text{for } c > 0.$$

Then

$$\delta_u := E(u) - E_{min} = cu^2, \quad \text{dist} = |u|, \quad \|\nabla E(u)\| = 2c|u|.$$

so we have Łojasiewicz estimates

$$|E(u) - c^*| \leq C\|E(u)\|^2 \qquad \text{for critical value } c^* = E(0) = 0$$

$$\|u - u^*\| \leq C\|\nabla E(u)\|, \qquad \text{for critical point } u^* = 0$$

and quantitative stability estimate

$$\|u - u^*\| \leq C\delta_u^{\frac{1}{2}}, \qquad \text{for minimiser } u^* = 0$$

Intuition:

In "nice non-degenerate" situations, we might hope for Łojasiewicz estimates of the form

$$
\begin{aligned}
|E(u) - c^*| &\leq C\|E(u)\|^{\gamma_1} && \text{for some } c^* \in \Xi_E := \{E(u) : u \in \mathcal{C}\} \\
\|u - u^*\| &\leq C\|\nabla E(u)\|^{\gamma_2}, && \text{for some } u^* \in \mathcal{C} := \{u : \nabla E(u) = 0\}
\end{aligned}
$$

for exponents $\gamma_1 = 2$ and $\gamma_2 = 1$, and quantitative stability estimate

$$
\|u - u^*\| \leq C\delta_u^{\gamma_3}, \qquad \text{for some } u^* \in \mathcal{M} := \{u : E(u) = \min E\}
$$

for $\gamma_3 = \frac{1}{2}$.

Intuition:

In "nice non-degenerate" situations, we might hope for Łojasiewicz estimates of the form

$$|E(u) - c^*| \leq C\|E(u)\|^{\gamma_1} \qquad \text{for some } c^* \in \Xi_E := \{E(u) : u \in \mathcal{C}\}$$
$$\|u - u^*\| \leq C\|\nabla E(u)\|^{\gamma_2}, \qquad \text{for some } u^* \in \mathcal{C} := \{u : \nabla E(u) = 0\}$$

for exponents $\gamma_1 = 2$ and $\gamma_2 = 1$, and quantitative stability estimate

$$\|u - u^*\| \leq C\delta_u^{\gamma_3}, \qquad \text{for some } u^* \in \mathcal{M} := \{u : E(u) = \min E\}$$

for $\gamma_3 = \frac{1}{2}$.
If there are null-directions of d^2E which do not correspond to tangential directions to $\mathcal{C}$ resp. $\mathcal{M}$ and E is analytic then we might hope for such estimates with smaller exponents.

Intuition:

In "nice non-degenerate" situations, we might hope for Łojasiewicz estimates of the form

$$|E(u) - c^*| \leq C\|E(u)\|^{\gamma_1} \qquad \text{for some } c^* \in \Xi_E := \{E(u) : u \in \mathcal{C}\}$$
$$\|u - u^*\| \leq C\|\nabla E(u)\|^{\gamma_2}, \qquad \text{for some } u^* \in \mathcal{C} := \{u : \nabla E(u) = 0\}$$

for exponents $\gamma_1 = 2$ and $\gamma_2 = 1$, and quantitative stability estimate

$$\|u - u^*\| \leq C\delta_u^{\gamma_3}, \qquad \text{for some } u^* \in \mathcal{M} := \{u : E(u) = \min E\}$$

for $\gamma_3 = \frac{1}{2}$.
If there are null-directions of d^2E which do not correspond to tangential directions to $\mathcal{C}$ resp. $\mathcal{M}$ and E is analytic then we might hope for such estimates with smaller exponents.

- True for PDEs and Variational problems with good convexity properties if there are **NO SINGULARITIES**, thanks to work of Łojasiewicz ('70s, finite dim) and Leon Simon ('83)

Dirichlet Energy and Harmonic maps

Let

- (Σ, g) be a closed surface
- (N, g_N) a closed Riemannian manifold (any dimension)

A map $u : \Sigma \to N$ is harmonic if it is a critical point of

$$E(u) = \frac{1}{2} \int_\Sigma |\nabla u|^2, \text{ i.e. so that } \tau_g(u) := -\nabla^{L^2} E(u) = 0.$$

Dirichlet Energy and Harmonic maps

Let

- (Σ, g) be a closed surface
- (N, g_N) a closed Riemannian manifold (any dimension)

A map $u : \Sigma \to N$ is harmonic if it is a critical point of

$$E(u) = \frac{1}{2}\int_\Sigma |\nabla u|^2, \text{ i.e. so that } \tau_g(u) := -\nabla^{L^2} E(u) = 0.$$

Recall:

- tension $\tau_g(u) = \text{tr}(\nabla du) = \Delta_g u + A(u)(\nabla u, \nabla u)$ (for $N \hookrightarrow \mathbb{R}^M$)
- E is conformally invariant wrt domain metric
- If u is (weakly) conformal and harmonic then it is a (branched) minimal immersion
- For $\Sigma = S^2$: Non-constant u is harmonic iff it is a (weakly) conformal parametrisation of a (branched) minimial sphere

Energy spectrum of harmonic maps

Conjecture (L. Simon '90s, F.H. Lin '99)

The energy spectrum

$$\Xi_E((\Sigma, g), N) := \{E(u) : u : (\Sigma, g) \to N \text{ harmonic, not constant}\}$$

is discrete for every closed analytic manifold N.

Energy spectrum of harmonic maps

Conjecture (L. Simon '90s, F.H. Lin '99)

The energy spectrum

$$\Xi_E((\Sigma, g), N) := \{E(u) : u : (\Sigma, g) \to N \text{ harmonic, not constant}\}$$

is discrete for every closed analytic manifold N.

For $\Sigma = S^2$ equivalent to: Areas of minimal spheres in N are discrete

Energy spectrum of harmonic maps

Conjecture (L. Simon '90s, F.H. Lin '99)

The energy spectrum

$$\Xi_E((\Sigma,g),N) := \{E(u) : u : (\Sigma,g) \to N \textit{ harmonic, not constant}\}$$

is discrete for every closed analytic manifold N.

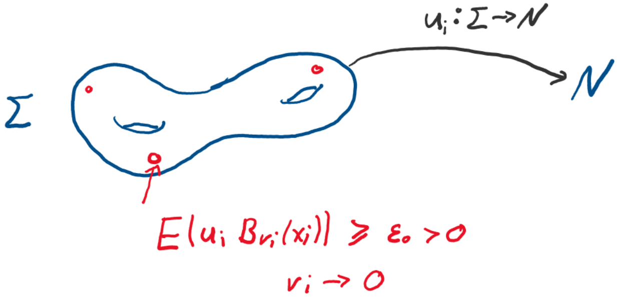
For $\Sigma = S^2$ equivalent to: Areas of minimal spheres in N are discrete

Known back then:

- True for very special cases where set of harmonic maps is known, e.g. $\Xi_E(S^2,S^2) = \{4\pi k, k \in \mathbb{N}\}$.
- For $\Sigma = S^2$: $E_N^* := \inf \Xi(S^2,N) > 0$ is achieved and isolated
- Any potential accumulation point of Ξ_E needs to have the form $E(u^*) + \sum_i E(\omega_i)$ for $u^* : \Sigma \to N$ and $\omega_i : S^2 \to N$ harmonic

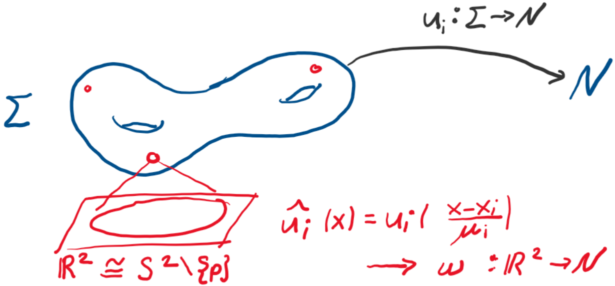
Formation of bubble trees: Compactness theory from 90's:

- Sequences of almost harmonic maps u_i, i.e. u_i with $\|\tau(u_i)\|_{L^2} \to 0$, converge smoothly to a limit u_∞ away from points where energy concentrates
- any concentration of energy corresponds to formation of (at least one) bubble, i.e. a highly scaled copy $\omega(\mu x)$ of a harmonic map $\omega : \mathbb{R}^2 \to N$.



Formation of bubble trees: Compactness theory from 90's:

- Sequences of almost harmonic maps u_i, i.e. u_i with $\|\tau(u_i)\|_{L^2} \to 0$, converge smoothly to a limit u_∞ away from points where energy concentrates
- concentration of energy must be caused by bubbling off of one or more minimial sphere(s)



Formation of bubble trees: Compactness theory from 90's:

- Sequences of almost harmonic maps u_i, i.e. u_i with $\|\tau(u_i)\|_{L^2} \to 0$, converge smoothly to a limit u_∞ away from points where energy concentrates
- concentration of energy must be caused by bubbling off of one or more minimial sphere(s)

Remark:
If $\Sigma = S^2$ and "all the energy concentrates at one scale and near one point", i.e. if $u_\infty = const$ and only one bubble forms, then we can pull-back by Möbius transforms to get smooth convergence.

Lowest candidate for accumulation point

First energy level where smooth convergence fails and hence result of Simon does not apply:

- For $\Sigma = S^2$:
$$2E_N^* = 2\inf \Xi_E(S^2, N)$$
corresponding to sequences of maps that converge to a bubble tree with non-trivial base map and a single bubble
- For $\Sigma \neq S^2$:
$$E_N^* = \inf \Xi_E(S^2, N)$$
corresponding to a sequence of maps that converges to a constant away from a single point where a bubble forms

Results on energy spectrum:

Theorem (R.'21)

Let $\Sigma \neq S^2$ be any surface, and let N be any analytic manifold for which the harmonic spheres with minimal energy are not branched.
Then E_N^* is not an accumulation point of $\Xi_E(\Sigma, N)$.

Results on energy spectrum:

Theorem (R.'21)

Let $\Sigma \neq S^2$ be any surface, and let N be any analytic manifold for which the harmonic spheres with minimal energy are not branched.
Then E_N^* is not an accumulation point of $\Xi_E(\Sigma, N)$.

Theorem (R.'23)

Let N be any analytic 3 manifold for which the harmonic spheres with minimal energy are not branched and non-degenerate.
Then $2E_N^*$ is not an accumulation point of $\Xi(\Sigma, N)$.

Results on energy spectrum:

Theorem (R.'21)

Let $\Sigma \neq S^2$ be any surface, and let N be any analytic manifold for which the harmonic spheres with minimal energy are not branched.
Then E_N^* is not an accumulation point of $\Xi_E(\Sigma, N)$.

Theorem (R.'23)

Let N be any analytic 3 manifold for which the harmonic spheres with minimal energy are not branched and non-degenerate.
Then $2E_N^*$ is not an accumulation point of $\Xi(\Sigma, N)$.

Get additional results on:

- Łojasiewicz estimates for maps that are close to corresponding bubble trees
- Convergence of harmonic map flow in which such singularities form
- Results on gluing of two minimal spheres

Basic idea of proof (Malchiodi-R.-Sharp '20)

- Construct a finite dimensional manifold $\mathcal{Z}$ of singularity models that are obtained by gluing a highly concentrated bubble onto a base map
- Show that such singularity models can be constructed so that
  - d^2E is uniformly definite orthogonal to $\mathcal{Z}$, i.e. so that the Jacobi-operator on $T_z^\perp\mathcal{Z}$ has a uniform eigenvalue gap around $\lambda = 0$
  - For each $z \in \mathcal{Z}$ which is not a critical point of E there is a unit direction $y_z \in T_z\mathcal{Z}$ with

$$dE(z)(y_z) \gg \|dE(z)\|^2 + \|dE(z)\| \cdot \|d^2E(z)(y_z, \cdot)\|$$

- Get Łojasiewicz estimates for free which tell us that

$$|E(u) - \bar{E}| \leq C\|\tau(u)\|_{L^2}^\gamma \text{ for } \bar{E} = E_N^* \text{ resp. } \bar{E} = 2E_N^*$$

  for any map u that is close to such a bubble tree. In particular $E(u) = \bar{E}$ if u is harmonic so Ξ_N can't accumulate at $\bar{E}$.

Rigidity of minimisers?

Suppose that $E_{min} = \text{inj}_{\mathcal{A}} E$ is achieved in some homotopy class $\mathcal{A}$ of maps $u : \Sigma \to N$ and let $\mathcal{M}$ be the set of minimisers.

Question: Do almost minimisers, i.e. maps with small energy defect

$$\delta_u = E(u) - E_{min}$$

"look like minimisers"?

Rigidity of minimisers?

Suppose that $E_{min} = \text{inj}_{\mathcal{A}} E$ is achieved in some homotopy class $\mathcal{A}$ of maps $u : \Sigma \to N$ and let $\mathcal{M}$ be the set of minimisers.

Question: Do almost minimisers, i.e. maps with small energy defect

$$\delta_u = E(u) - E_{min}$$

"look like minimisers"? I.e.

Does

$$\delta_{u_i} \to 0 \text{ imply } \text{dist}_{H^1}(u_i, \mathcal{M}) \to 0$$

and can we bound

$$\text{dist}^2_{H^1}(u, \mathcal{M}) \leq C\delta_u^{\alpha}$$

Rigidity of minimisers for maps from S^2 to S^2

In the special case of maps $v : S^2 \to S^2$ we have

$$E(v) \geq 4\pi |\deg(v)|$$

with " = " iff v is a rational map, i.e. given by a meromorphic function

$$\hat{v} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$$

in either z or $\bar{z}$ in stereographic coordinates on both the domain and target.

Rigidity of minimisers for maps from S^2 to S^2

In the special case of maps $v : S^2 \to S^2$ we have

$$E(v) \geq 4\pi |\deg(v)|$$

with " $=$ " iff v is a rational map, i.e. given by a meromorphic function

$$\hat{v} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$$

in either z or $\bar{z}$ in stereographic coordinates on both the domain and target.

Question:
Can we bound the distance of maps v with degree k from the set of degree k rational maps by the energy defect

$$\delta_v := E(v) - 4\pi |\deg(v)|?$$

Rigidity for degree ± 1 maps

Theorem

For any map $v : S^2 \to S^2$ of degree ± 1 there exists a Möbius transform ω (in z or $\bar{z}$) so that

$$\int_{S^2} |\nabla(v-\omega)|^2 dv_g \leq C\delta_v.$$

- Due to Bernand-Mantel, Muratov and Simon ('19) in paper on ultra-thin ferromagnetic films and skyrmions
- Simplified proofs by Topping ('20) and by Hirsch and Zemas ('20)
- Extension to higher dimensions: Guerra, Lamy, Zemas ('23)

Rigidity for degree ± 1 maps

Theorem

For any map $v : S^2 \to S^2$ of degree ± 1 there exists a Möbius transform ω (in z or $\bar{z}$) so that

$$\int_{S^2} |\nabla(v - \omega)|^2 dv_g \leq C\delta_v.$$

- Due to Bernand-Mantel, Muratov and Simon ('19) in paper on ultra-thin ferromagnetic films and skyrmions
- Simplified proofs by Topping ('20) and by Hirsch and Zemas ('20)
- Extension to higher dimensions: Guerra, Lamy, Zemas ('23)

Key aspect of degree 1 maps:

- Energy can only concentrate at a single point and scale
- $\{$minimisers $\} = \{$ symmetry group $\}$

Rigidity for maps of general degree?

- Deng-Sun-Wei ('21): Local result (where energy concentration is a priori excluded) and conjecture of global quantitative rigidity estimate

$$\mathrm{dist}^2_{H^1}(v, \mathcal{M}_k) \leq C\delta_v |\log \delta_v|, \quad \delta_v = E(v) - 4\pi k$$

for

$$\begin{aligned}\mathcal{M}_k &= \{\text{degree } k \text{ rational maps}\} \\ &= \{\tfrac{p(z)}{q(z)} : \text{ polynomials with } \max(\deg(p), \deg(q)) = k\}\end{aligned}$$

Rigidity for maps of general degree?

- Deng-Sun-Wei ('21): Local result (where energy concentration is a priori excluded) and conjecture of global quantitative rigidity estimate

$$\mathrm{dist}^2_{H^1}(v,\mathcal{M}_k) \le C\delta_v|\log\delta_v|, \quad \delta_v = E(v) - 4\pi k$$

for

$$\begin{aligned}\mathcal{M}_k &= \{\text{degree } k \text{ rational maps}\} \\ &= \{\tfrac{p(z)}{q(z)} : \text{ polynomials with } \max(\deg(p),\deg(q)) = k\}\end{aligned}$$

BUT as $E(v) \ge 4\pi k = 4\pi + \ldots + 4\pi$,

Rigidity for maps of general degree?

- Deng-Sun-Wei ('21): Local result (where energy concentration is a priori excluded) and conjecture of global quantitative rigidity estimate

$$\mathrm{dist}^2_{H^1}(v, \mathcal{M}_k) \leq C\delta_v |\log \delta_v|, \quad \delta_v = E(v) - 4\pi k$$

for

$$\begin{aligned} \mathcal{M}_k &= \{\text{degree } k \text{ rational maps}\} \\ &= \{\tfrac{p(z)}{q(z)} : \text{ polynomials with } \max(\deg(p), \deg(q)) = k\} \end{aligned}$$

BUT as $E(v) \geq 4\pi k = 4\pi + \ldots + 4\pi$,

- Energy can concentrate at multiple scales/points
- Maps that describe behaviour on different "microdomains" and/or bulk might not match up as it is "cheap" to transition between different values on annuli with degenerating conformal structures

Rigidity for maps of general degree? NO!

Rigidity of minimisers is WRONG even at qualitative level for any degree $k \geq 2$ as there exist $v_n : S^2 \to S^2$ of any degree $k \geq 2$ with

$$\delta_{v_n} \to 0 \text{ but } \operatorname{dist}(v_n, \mathcal{M}_k\}) \nrightarrow 0.$$

Rigidity for maps of general degree? NO!

Rigidity of minimisers is WRONG even at qualitative level for any degree $k \geq 2$ as there exist $v_n : S^2 \to S^2$ of any degree $k \geq 2$ with

$$\delta_{v_n} \to 0 \text{ but } \operatorname{dist}(v_n, \mathcal{M}_k\}) \nrightarrow 0.$$

Counterexample (k=2, in stereographic coordinates):
Modify $\omega(z) = z + \frac{1}{\mu z}$, $\mu \gg 1$ by shifting by a constant $a \neq 0$ on the microdomain and using "cheap" way to transition.
Resulting map v_μ with

$$\begin{aligned} v_\mu(z) &\approx z \text{ away from } 0 \\ v_\mu(z) &\approx \frac{1}{\mu z} + a \text{ for } |z| \leq C\mu^{-1} \end{aligned} \tag{1}$$

will be so that

- Distance from any rational map is $\gtrsim |a|$
- $\delta_{v_n}^2 \sim \frac{|a|^2}{\log \mu} \to 0$

Rigidity for maps of general degree? Maybe...

Better (?) Question:

- Is every map v with small energy defect essentially described by a collection of rational maps that describe the behaviour of v on bulk and on different "microdomains"?
- If so, can we bound the distance of v to such a collection of rational maps in terms of the energy defect?

Distance of a single map to a collections of maps?

Let $\omega_1, \ldots, \omega_j$ be a collection of rational maps from S^2 to S^2.
These maps correspond to very different scales/concentration points if there is a partition of S^2 into sets Ω_i so that

- all Ω_i are obtained from balls by removing a finite number of balls with far smaller radius
- ω_i is essentially constant outside of Ω_i in the sense that
  - $\int_{S^2 \setminus \Omega_i} |\nabla \omega_i|^2$ and
  - the oscillation of ω_i over connected components of $S^2 \setminus \Omega_i$

  are very small.

Distance of a single map to a collections of maps?

Let $\omega_1, \ldots, \omega_j$ be a collection of rational maps from S^2 to S^2.
These maps correspond to very different scales/concentration points if there is a partition of S^2 into sets Ω_i so that

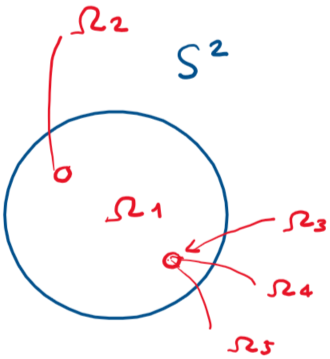
- all Ω_i are obtained from balls by removing a finite number of balls with far smaller radius
- ω_i is essentially constant outside of Ω_i in the sense that
  - $\int_{S^2 \setminus \Omega_i} |\nabla \omega_i|^2$ and
  - the oscillation of ω_i over connected components of $S^2 \setminus \Omega_i$

  are very small.

We can then say that a map v is close to such a collection of maps if

$$\sum_i \int_{\Omega_i} |\nabla (v - \omega_i)|^2 \text{ is very small .}$$

Distance of a single map to a collections of maps?



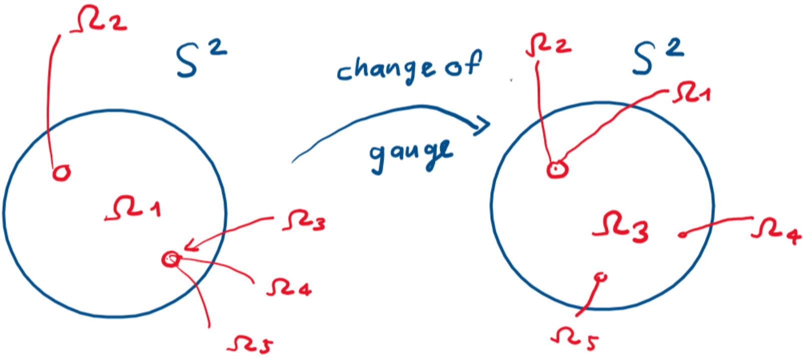
$v \approx w_1$

$w_j \approx \text{const.} \quad j \neq 1$

on bulk of domain

$S^2 \setminus {\circ}_{\circ}$

Distance of a single map to a collections of maps?



Distance of a single map to a collections of maps?

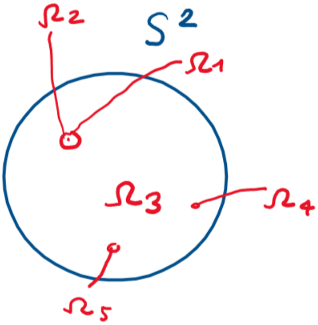
$v \approx w_3$

$w_j \approx \text{const} \quad j \neq 3$

on bulk of domain

$S^2 \setminus$

New gauge



Rigidity for maps of general degree? YES!

Theorem (R. '23)

Any degree k map $v : S^2 \to S^2$ with small energy defect is close to a collection of rational maps $\omega_1, \ldots, \omega_j$, $j \geq 1$, with total degree k in the sense that

$$\sum_i \int_{\Omega_i} |\nabla(v - \omega_i)|^2 \leq C\delta_v |\log \delta_v|$$

for a partition of S^2 into sets Ω_i as above which are so that

$$\int_{S^2 \setminus \Omega_i} |\nabla \omega_i|^2 \leq C\delta_v^{2\alpha} \text{ and } \operatorname*{osc}_v \omega_i \leq C\delta_v^{\alpha}.$$

Here $\alpha < \infty$ can be chosen as any number and $C = C(k, \alpha)$.

Rigidity for maps of general degree? YES!

Theorem (R. '23)

Any degree k map $v : S^2 \to S^2$ with small energy defect is close to a collection of rational maps $\omega_1, \ldots, \omega_j$, $j \geq 1$, with total degree k in the sense that

$$\sum_i \int_{\Omega_i} |\nabla(v - \omega_i)|^2 \leq C\delta_v |\log \delta_v|$$

for a partition of S^2 into sets Ω_i as above which are so that

$$\int_{S^2 \setminus \Omega_i} |\nabla \omega_i|^2 \leq C\delta_v^{2\alpha} \text{ and } \operatorname*{osc}_v \omega_i \leq C\delta_v^{\alpha}.$$

Here $\alpha < \infty$ can be chosen as any number and $C = C(k, \alpha)$.

This estimate is sharp!

I.e. there is no function that decays faster than $\phi(\delta) = \delta |\log \delta|$ for which the above estimate holds.

THANK YOU!

Ideas of proof: Simplest case

If energy is not concentrated on any small set $\Rightarrow$ Topping's proof from $\deg = \pm 1$ case applies, i.e.

- Evolve v by standard harmonic map flow

$$\partial_t u = \tau_{g_{S^2}}(u) \text{ with } u(0) = v$$

- Use that known Łojasiewicz estimate $\delta_u \leq C\|\tau_{g_{S^2}}(u)\|^2_{L^2(S^2, g_{S^2})}$ implies that

$$\|u(t) - u(0)\|_{L^2} \leq \int_0^t \|\partial_t u\|_{L^2} \leq C[E(t=0) - E(t)]$$

to deduce that flow remains smooth for all times and evolves map to a single rational map $\omega = u(\infty)$ with well controlled energy density and

$$\|\omega - v\|_{H^1} \leq C\delta_v.$$

Ideas of proof: Next simplest case

If energy is not too concentrated in the sense that all balls that contain a certain amount ε_1 of energy have radius at least

$$r \geq \delta_v^\alpha$$

then:

- Determine Möbius transforms M_i that scale up each such ball B_i to unit size
- Consider weighted metric $g = \sum M_i^* g_{S^2}$ and evolve v with weighted harmonic map flow
- Show a Łojasiewicz estimate that involves such weighted metrics
- Flow will be well controlled as energy is not concentrated (wrt g) and converges to a rational map ω that is close to v.

Ideas of proof: General case

If energy concentrates at very different scales:

- Partition domain into sets Ω_i that correspond to very different scales
- For each i, consider weighted metric that rescales all parts of Ω_i that contain ε_1 of energy to unit size
- **Flow** to rational map $\tilde{\omega}_i$ that approximates v well on Ω_i but could be very concentrated elsewhere
- **Cut out** highly concentrated parts to get new initial map that is still close to v on Ω_1
- **Flow again**