Mean curvature flow with generic initial data

Felix Schulze,
University of Warwick

Joint work with O. Chodosh (Stanford), K. Choi (KIAS, Seoul) and C. Mantoulidis (Rice)

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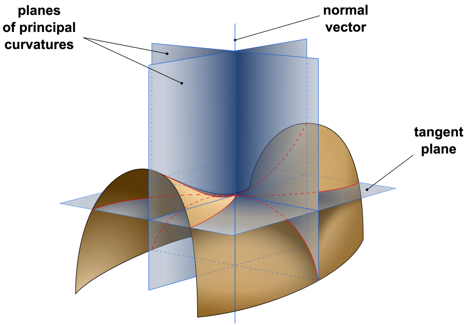
Background

Consider $(M^n(t))_{0\leq t<T}$ a smooth mean curvature flow of hypersurfaces in $\mathbb{R}^{n+1}$, i.e.

$$\left(\frac{\partial F}{\partial t}\right)^{\perp} = \vec{H} = -H\nu = \Delta_{M(t)} F$$

for a smooth family $F(\cdot, t)$ of parametrisations, and its space-time track

$$\mathcal{M} = \cup_{0\leq t<T} M_t \times \{t\} \subset \mathbb{R}^{n+1} \times \mathbb{R}\,.$$



Mean curvature $H = \lambda_1 + \cdots + \lambda_n$

(Source: Wikipedia)

Background

Basic properties:

- Gradient flow of area, geometric heat equation
- Quasi-linear parabolic: smooth short-time existence
- Avoidance principle: if $(M_1(t))_{0\leq t<T}$ and $(M_2(t))_{0\leq t<T}$ two solutions of mean curvature flow, then

$$M_1(0)\cap M_2(0)=\emptyset \implies M_1(t)\cap M_2(t)=\emptyset$$

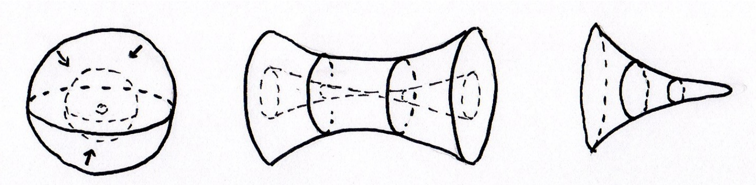
- Finite existence time $\rightsquigarrow$ singularities
- Convexity and mean convexity preserved
- Continuation through singularities as weak mean curvature flow, possibly non-unique

Background

Theorem (Gage-Hamilton ('86), Grayson ('87)): Curve shortening flow contracts a simple, closed curve in $\mathbb{R}^2$ in finite time to a 'round point'.

Theorem (Huisken ('84)): Mean curvature flow contracts a closed, convex hypersurface in $\mathbb{R}^{n+1}$ in finite time to a 'round point'.

Problem: Singularities



Applications of mean curvature flow through singularities

- Classification of 2-convex surfaces (Huisken-Sinestrari ('09), Haslhofer-Kleiner ('17), Brendle-Huisken ('16, '17))
- Topology of moduli space of 2-convex surfaces (Buzano-Haslhofer-Hershkovits ('16, '17))
- Constructing new minimal surfaces (Haslhofer-Ketover ('19))
- Schoenflies problem for low-entropy 3-spheres (Bernstein-Wang ('20), Choi-Chodosh-Mantoulidis-Schulze ('20,'21), Daniels-Holgate ('21))
- Isoperimetric inequalities (Hershkovits ('17), Schulze ('20))
- Structure of 3-d manifolds with positive scalar curvature (Liokumovich-Maximo ('20))

Monotonicity formula: backwards heat kernel based at $X_0 = (x_0, t_0)$:

$$\rho_{X_0}(x,t) = \frac{1}{(2\pi(t_0-t))^{n/2}} e^{-\frac{|x-x_0|^2}{4(t_0-t)}},$$

then

$$\frac{d}{dt}\int_{M_t} \rho_{X_0}\, d\mathcal{H}^n \leq -\int_{M_t} \left|\vec{H} + \frac{(x-x_0)^\perp}{2(t_0-t)}\right|^2 \rho_{X_0}\, d\mathcal{H}^n$$

Tangent flows: Consider $\mathcal{D}_\lambda : (x,t) \mapsto (\lambda x, \lambda^2 t)$ and $\lambda_i \to +\infty$, then subsequentially

$$\mathcal{D}_{\lambda_i}(\mathcal{M} - X_0) \rightharpoonup \mathcal{M}'.$$

and by the monotonicity formula $\mathcal{D}_\lambda(\mathcal{M}' \cap \{t<0\}) = \mathcal{M}' \cap \{t<0\}$, i.e.

$$\mathcal{M}'(t) = \sqrt{-t}\cdot\Sigma$$

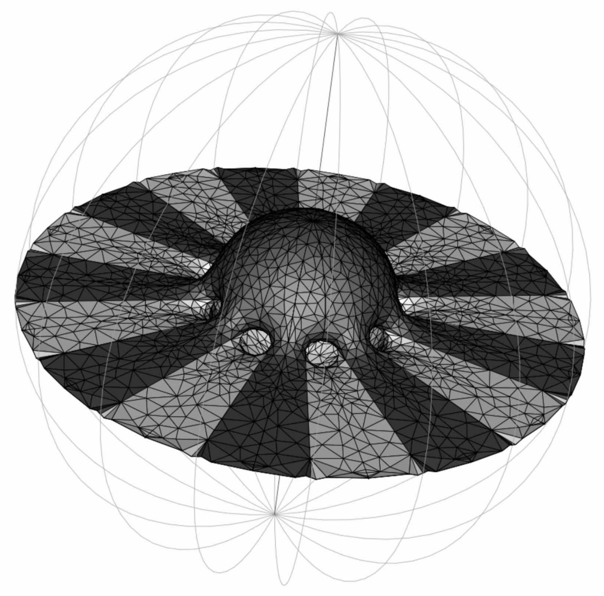
and Σ satisfies

$$\vec{H} = -\frac{x^\perp}{2}.$$

We call such a Σ a *self-shrinker*.

Examples:

- Plane: $\mathbb{R}^n \subset \mathbb{R}^{n+1}$
- Sphere: $\mathbb{S}^n_{\sqrt{2n}} \subset \mathbb{R}^{n+1}$
- (Generalized) cylinders: $\mathbb{S}^{n-k}_{\sqrt{2(n-k)}} \times \mathbb{R}^k \subset \mathbb{R}^{n+1}$ for $k = 1, \ldots, n-1$
- Huisken ('90): If $H \geq 0$ (which is preserved under the evolution), then these are the only possibilities.
- Angenent ('89): torus of revolution
- Kapouleas-Kleene-Møller ('11), X.H. Nguyen ('11): desingularisation of $\mathbb{R}^2 \cup \mathbb{S}^2_2$



Tom Ilmanen's conjectural shrinker of genus 8 with 9 Scherk handles

(picture used with his permission)

Monotonicity formula and tangent flows

Structure of self-shrinkers:

- $\lim_{\lambda \searrow 0} \lambda \cdot \Sigma = C_\infty$ asymptotic cone (as sets)
- We call Σ *asymptotically conical* if C_∞ and convergence smooth
- L. Wang ('16): $\Sigma^2 \subset \mathbb{R}^3$ embedded with finite genus $\Rightarrow \Sigma^2$ has only cylindrical or smoothly conical ends ('16)
- S. Brendle ('16): the only embedded genus zero shrinkers in $\mathbb{R}^3$ are the sphere and the cylinder

Generic singulartities

Fundamental issue:

Zoo of singularities, no hope of classification

Genericity principle:

Generic solutions, obtained by small perturbations of the initial data, exhibit simpler singularities.

Conjecture (Huisken):

A generic mean curvature flow in $\mathbb{R}^3$ has only spherical and cylindrical singularities

Colding-Minicozzi ('12):

- The only linearly stable singularity models are spheres and (generalised) cylinders

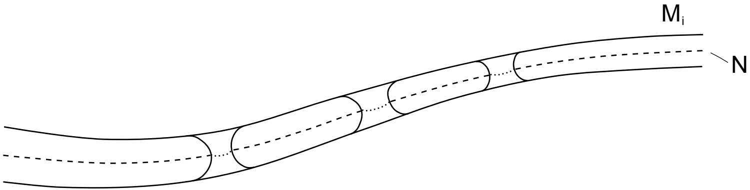
Question:

- How to perturb away unstable singularity models?
- Perturb only the initial condition, past singularities?

Perturbative results

Theorem 1 (CCMS ('20), CCS ('23)): *Let $M^\circ \subset \mathbb{R}^3$ be a closed embedded surface. There exist arbitrary small C^∞ graphs M over M° so that mean curvature flow starting at $M(0) := M$ has only spherical and cylindrical singularities for as long as its singularities have multiplicity one.*

Problem: Multiplcity



Convergence of the surfaces M_i with multiplicity two to the dotted surface N, while "necks" are pinching off.

Perturbative results

Theorem (Bamler – Kleiner ('23)): *For closed embedded surfaces $M(0) \subset \mathbb{R}^3$, mean curvature flow has only singularities with multiplicity one at the first non-generic time.*

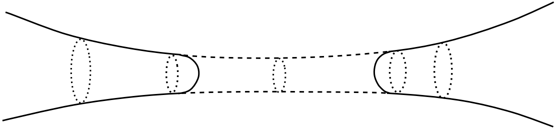
Corollary: *Let $M^\circ \subset \mathbb{R}^3$ be a closed embedded surface. There exist arbitrary small C^∞ graphs M over M° so that mean curvature flow starting at $M(0) := M$ has only multiplicity one spherical and cylindrical singularities.*

Remarks:

- A (weak) mean curvature flow with only multiplcity one generic singularities is unique.
- The space of (weak) mean curvature flows with only multiplicity one generic singularities is open. Thus the set of M in the theorem above is both dense and open.

Flows with surgery

Surgery:



- Close to a cylindrical singularity, replacing a cylindrical piece by two spherical caps.
- Surgery for mean curvature flow of 2-convex surfaces (Huisken-Sinestrari ('09), Haslhofer-Kleiner ('17), Brendle-Huisken ('16, '17))

Theorem (Daniels-Holgate ('21)): *Any (weak) mean curvature flow with only spherical and cylindrical singularities starting from a smooth closed embedded hypersurface $M^2 \subset \mathbb{R}^3$ can be approximated by smooth flows with surgery.*

Corollary: *Let $M^\circ \subset \mathbb{R}^3$ be a closed embedded surface. There exist arbitrary small C^∞ graphs M over M° and a smooth mean curvature flow with surgery starting from M.*

Strategy of proof of Theorem 1

- Consider $M_0 \subset \mathbb{R}^{n+1}$ a fixed hypersurface, $\mathcal{M}_0$ a weak mean curvature flow starting at M_0.
- Consider a foliation $\{M_s\}_{s\in(-1,1)}$ around M_0. Embedd the flow $\mathcal{M}_0$ into a family of (weak) flows $\mathcal{M}_s$ starting at M_s.
- Avoidance principle: $\mathcal{M}_s(t) \cap \mathcal{M}_{s'}(t) = \emptyset$ for $s \neq s'$.
- Consider (x_0, t_0) a singular point of $\mathcal{M}_0$ and $\lambda_i \to \infty$ such that $\mathcal{D}_{\lambda_i}(\mathcal{M}_0 - (x_0, t_0)) \rightharpoonup \mathcal{M}'$, a tangent flow at X.
- Pass the whole foliation to the limit simultaneously, i.e. consider the flows $\mathcal{D}_{\lambda_i}(\mathcal{M}_s - (x_0, t_0))$ as $\lambda_i \to \infty$.
- Choosing $s_i \searrow 0$ carefully as $\lambda_i \to \infty$, up to a subsequence, $\mathcal{D}_{\lambda_i}(\mathcal{M}_{s_i} - (x_0, t_0))$ will converge to a non-empty flow $\overline{\mathcal{M}}$ that stays on one side of the original tangent flow $\mathcal{M}'$ and is ancient.
- Show that $\overline{\mathcal{M}}$ is unique up to parabolic scaling, moves in a rescaled sense in one direction $\Rightarrow$ thus has only spherical and cylindrical singularities and has genus zero near $(0, 0)$.
- Use this to find a choice of s small so that $\mathcal{M}_s$ has only spherical and cylindrical singularities near (x_0, t_0) and strictly drops genus.
- Iterate this.

Entropy (Colding-Minicozzi)

$$\lambda(M) := \sup_{x_0 \in \mathbb{R}^{n+1}, t_0 > 0} \int_M (4\pi t_0)^{-n/2} e^{-\frac{|x-x_0|^2}{4t_0}} \, d\mu$$

- $t \mapsto \lambda(M_t)$ monotonically decreasing under mean curvature flow

Classification of surfaces of low entropy:

- $\lambda(\mathbb{S}^n) = F(\mathbb{S}^n_{\sqrt{2n}}) < \lambda(\mathbb{S}^{n-1} \times \mathbb{R}) < \cdots < \lambda(\mathbb{S}^1 \times \mathbb{R}^{n-1}) = F(\mathbb{S}^1_{\sqrt{2}})$
- Colding-Ilmanen-Minicozzi-White ('13): the round sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ has lowest entropy among non-planar self-shrinkers.
- Bernstein-Wang ('16) (Zhu ('20)): the round sphere has lowest entropy among all closed hypersurfaces.
- Bernstein-Wang ('18): $\mathbb{S}^1 \times \mathbb{R}$ has second least entropy among non-planar self-shrinkers in $\mathbb{R}^3$.
- Bernstein-Wang ('18): $M^3 \subset \mathbb{R}^4$ closed and $\lambda(M) \leq \lambda(\mathbb{S}^2 \times \mathbb{R}) \Rightarrow M$ diffeomorphic to $\mathbb{S}^3$.

Perturbative results with low entropy

Theorem 2 (CMS ('23)): *If $M^3 \subset \mathbb{R}^4$ is a closed embedded hypersurface with entropy $\lambda(M) \leq 2$ then there exist arbitrarily small C^∞ graphs M' over M such that the mean curvature flow starting from M' has only multiplicity-one singularities of $\mathbb{S}^3, \mathbb{S}^2 \times \mathbb{R}$, and $\mathbb{S}^2 \times \mathbb{R}^2$-type.*

Remark:

- In this low entropy setting we can work globally in space-time, and without the classification of ancient one sided flows.This is replaced by a (softer) infinitesimal version for tangent flows, together with separation estimates (i.e. growth estimates for Jacobi fields) and a refined covering argument.
- Geometric measure theory yields that area minimizing hypersurfaces $M^n \subset \mathbb{R}^{n+1}$ are smooth for $n+1 \leq 7$ and for $n+1 \geq 8$ have a singular set of dimension at most $n-7$. With similar techniques we have been able to show that area minimizing hypersurfaces in $\mathbb{R}^{n+1}$ for $n+1 = 8, 9, 10$ are generically smooth (for $n+1 = 8$ this is originally due to Hardt–Simon ('85)).

Low entropy Schoenflies theorem

Schoenflies conjecture:

Any smoothly embedded $S^3 \subset \mathbb{R}^4$ bounds a smooth 4-ball.

Corollary (CCMS ('20, '21)): *If $M^3 \subset \mathbb{R}^4$ has entropy $\lambda(M) \leq \lambda(\mathbb{S}^2 \times \mathbb{R})$, then after a small C^∞-perturbation to a nearby hypersurface M', the mean curvature flow $M'(t)$ is completely smooth until it disappears in a round point.*

This yields an alternate proof of the low entropy Schoenflies theorem of Bernstein-Wang:

Theorem (Bernstein-Wang ('20)): *If $M^3 \subset \mathbb{R}^3$ has entropy $\lambda(M) \leq \lambda(\mathbb{S}^2 \times \mathbb{R})$ then M is smoothly isotopic to the round $\mathbb{S}^3$.*

Combining Theorem 2 with the approximation result by Daniels-Holgate we can improve this further:

Corollary (CCMS, Daniels-Holgate ('21)): *Any smoothly embedded $M^3 \subset \mathbb{R}^4$ which is homeomorphic to $\mathbb{S}^3$ and has entropy $\lambda(M) \leq \lambda(\mathbb{S}^1 \times \mathbb{R}^2)$ is smoothly isotopic to the round $\mathbb{S}^3$.*

Thank you!