Stability aspects of the Möbius group of $\mathbb{S}^{n-1}$ and bubbles of the (2-dim) H-system

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Talk based on the following works:

- S. Luckhaus, K. Zemas. *Rigidity estimates for isometric and conformal maps from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$*, Inventiones Mathematicae 230(1) (2022), 375–461.

- J. Hirsch, K. Zemas. *A note on a rigidity estimate for degree ± 1 conformal maps on $\mathbb{S}^2$*. Bulletin of the London Mathematical Society 54(1) (2022), 256–263.

- A. Guerra, X. Lamy, K. Zemas. *Sharp quantitative stability of the Möbius group among sphere-valued maps in arbitrary dimension*, to appear in Transactions of the American Mathematical Society (2024), arXiv: 2305.19886.

- A. Guerra, X. Lamy, K. Zemas. *Optimal Quantitative Stability of the Möbius group of the sphere in all dimensions*, under review (2024), arXiv: 2401.06593.

- A. Guerra, X. Lamy, K. Zemas. *On the existence of degenerate solutions of the two-dimensional H-system* (2024), arXiv: 2409.18068

Liouville's rigidity theorem as a differential inclusion

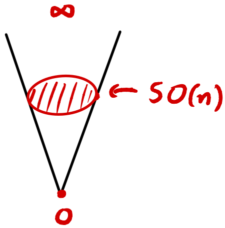
◇ **(For isometries)**

Let $n \geq 2$, $U \subset \mathbb{R}^n$ be a bounded Lipschitz domain. If $u \in W^{1,2}(U;\mathbb{R}^n)$ is s.t.

$$\nabla u \in SO(n) \ \text{ a.e. in } U\,,$$

then u is a rigid motion, i.e., $u(x) = Rx + b$, where $R \in SO(n)\,, b \in \mathbb{R}^n$.

◇ **(For conformal maps)**



Let $n \geq 3$, U as above. If $u \in W^{1,n}(U;\mathbb{R}^n)$ is s.t.

$$\nabla u \in \mathbb{R}_+ SO(n) \ \text{ a.e. in } U\,,$$

then u is a Möbius map, i.e.,

$$u(x) = AB\frac{x-a}{|x-a|^{\gamma}} + b\,,$$

where $\gamma = 0$ or 2, $A \in \mathbb{R}_+ SO(n)$, $B = \mathrm{diag}(1,\ldots,-1)$, $a \in \mathbb{R}^n \setminus U\,, b \in \mathbb{R}^n$.

Liouville (C^3), Reshetnyak ($W^{1,n}$), Iwaniec ($W^{1,p}$, $\frac{n}{2} \leq p_n \leq p \leq n$), Iwaniec-Martin (sharp $p_n = \frac{n}{2}$ for n even, same is conjectured for n odd!)

Liouville's theorem for conformal maps on $\mathbb{S}^{n-1}$, $n \geq 3$

An orientation preserving/reversing $u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$ of degree 1/-1 is generalized conformal, i.e.,

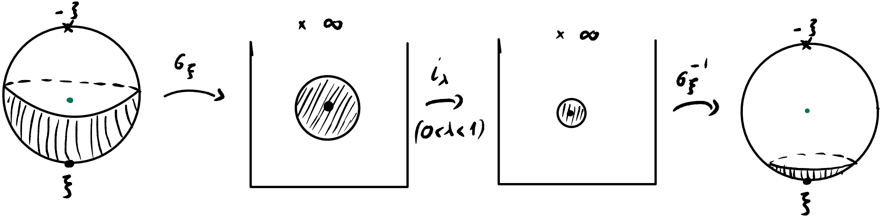
$$\nabla_T u^t \nabla_T u = \frac{|\nabla_T u|^2}{n-1} I_x \quad \mathcal{H}^{n-1}\text{-a.e. on } \mathbb{S}^{n-1},$$

iff it is a Möbius transformation of $\mathbb{S}^{n-1}$, i.e.,

$$u = O\phi_{\xi,\lambda} := O(\sigma_\xi^{-1} \circ i_\lambda \circ \sigma_\xi)$$

for some $O \in O(n)$, $\xi \in \mathbb{S}^{n-1}$ and $\lambda > 0$.

Here, σ_ξ is the stereographic projection from $-\xi \in \mathbb{S}^{n-1}$ onto $T_\xi \mathbb{S}^{n-1} \cup \{\infty\}$, and i_λ is the dilation in $T_\xi \mathbb{S}^{n-1}$ by factor $\lambda > 0$.



New proof on $\mathbb{S}^{n-1}$ (G.O.P. conformal, deg=1)

- Given $u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$ of degree 1, $\exists\, \phi_{\xi_0,\lambda_0}$: $\fint_{\mathbb{S}^{n-1}} u \circ \phi_{\xi_0,\lambda_0} = 0$.

- The map $\tilde{u} := u \circ \phi_{\xi_0,\lambda_0}$ of mean value 0, is also G.O.P.C. of degree 1.

- By conformality of $\tilde{u}$,

$$\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T \tilde{u}|^2}{n-1}\right)^{\frac{n-1}{2}} = \fint_{\mathbb{S}^{n-1}} \tilde{u}^\sharp(dv_g) = \deg(\tilde{u}) = 1\,,$$

hence,

$$1 = \fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T \tilde{u}|^2}{n-1}\right)^{\frac{n-1}{2}} \overset{\text{(Jensen)}}{\geq} \left(\fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T \tilde{u}|^2}{n-1}\right)^{\frac{n-1}{2}} \overset{\text{(Poincaré)}}{\geq} \left(\fint_{\mathbb{S}^{n-1}} |\tilde{u}|^2\right)^{\frac{n-1}{2}} = 1\,.$$

- Equality in the sharp L^2-Poincaré on $\mathbb{S}^{n-1} \implies \tilde{u}(x) = Rx$ for $R \in \mathbb{R}^{n\times n}$ (via expansion in **spherical harmonics**).

- Since $\tilde{u}(\mathbb{S}^{n-1}) = \mathbb{S}^{n-1}$ and $\deg(\tilde{u}) = 1$, we deduce that $R \in SO(n)$.

Theorem (An optimal quantitative extension for $\mathbb{S}^{n-1}$-valued maps)

For every $u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$ (with $\deg u := \fint_{\mathbb{S}^{n-1}} \langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \rangle = 1$),

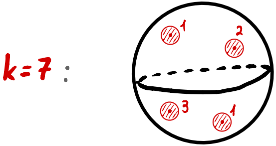
$$\inf_{\phi \in Mob_+(\mathbb{S}^{n-1})} \fint_{\mathbb{S}^{n-1}} |\nabla_T u - \nabla_T \phi|^{n-1} \lesssim \left(\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u|^2}{n-1} \right)^{\frac{n-1}{2}} - 1 \right).$$

- $n = 3$: Mantel-Muratov-Simon ('21, ARMA), Hirsch-Z. ('22, Bull. of LMS), Topping ('23, Bull. of LMS).

- Maps of degree $k \geq 2$: Rupflin ('23). The optimal estimate is of the form

$$\mathrm{dist}^2_{W^{1,2}}(u, \mathcal{R}) \lesssim \delta_u(|\log(\delta_u)| + 1), \ \delta_u := \frac{1}{2} \fint_{\mathbb{S}^2} |\nabla_T u|^2 - k \geq 0,$$

and $\mathcal{R}$ describes collections of rational maps at very different scales.

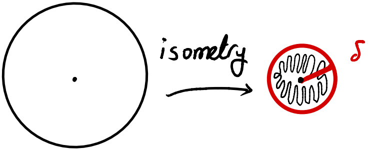


- $n \geq 4$: Guerra-Lamy-Z. ('24, TAMS).

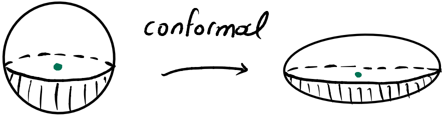
Flexibility vs Rigidity of Isometric and Conformal maps from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$

- **Classical rigidity in the Weyl problem for isometric embeddings**: The only C^2 isometric embeddings of $\mathbb{S}^{n-1}$ into $\mathbb{R}^n$ are rigid motions.

- **Flexibility via the celebrated Nash-Kuiper theorem**: For every arbitrarily small ball B_δ, there exist C^1 isometric embeddings wrinkling $\mathbb{S}^{n-1}$ inside B_δ.



- For conformal maps from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$, other examples that are not Möbius are also (when $n = 3$) used in cartography (Jacobi's conformal map projection), others are provided by the Uniformization Theorem, . . .



- **Liouville's rigidity theorem on** $\mathbb{S}^{n-1}$ on one hand, and the above **flexibility phenomena** on the other, indicate that an **extra deficit** for the deviation of $u(\mathbb{S}^{n-1})$ from being a round sphere is necessary for the stability of its isometry (resp. conformal) group among low regularity maps from $\mathbb{S}^{n-1}$ into $\mathbb{R}^n$.

Stability in the conformal case, $n \geq 3$

If $u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{R}^n)$, then

$$\underbrace{\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u|^2}{n-1}\right)^{\frac{n-1}{2}}}_{=:D_{n-1}(u)} \overset{(A.M.-G.M.)}{\geq} \fint_{\mathbb{S}^{n-1}} \sqrt{\det\left(\nabla_T u^t \nabla_T u\right)} \overset{(I.I.)}{\geq} \Bigg| \underbrace{\fint_{\mathbb{S}^{n-1}} \langle u, \overbrace{\bigwedge_{i=1}^{n-1} \partial_{\tau_i} u}^{=:J(u)} \rangle}_{=:V_n(u)} \Bigg|^{\frac{n-1}{n}} .$$

- ◇ "=" in A.M.-G.M. iff u is generalized conformal from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$.
- ◇ "=" in I.I. iff $u(\mathbb{S}^{n-1}) \subset \partial B_r(x_0)$ $\mathcal{H}^{n-1}$-a.e. on $\{J(u) \neq 0\}$; $r > 0, x_0 \in \mathbb{R}^n$.
- ◇ "=" in both $\implies$ u is a conformal solution to the H-**system**

$$\Delta_{n-1} u + H_u J(u) = 0 \text{ on } \mathbb{S}^{n-1}, \quad H_u := (n-1)^{\frac{n-1}{2}} \frac{D_{n-1}(u)}{V_n(u)},$$

hence $C^{1,\alpha}$ (Mou-Yang '96, J. Geom. Anal.) $\Longrightarrow$ ... modulo translation and rescaling is a conformal self-map of $\mathbb{S}^{n-1}$ of degree ± 1, i.e., is Möbius.

◇ Thus, the quantity

$$\mathcal{E}_{n-1}(u) := \frac{[D_{n-1}(u)]^{\frac{n}{n-1}}}{|V_n(u)|} - 1 \geq 0\,,$$

provides the correct deficit for stability of the Möbius group of $\mathbb{S}^{n-1}$ among maps into $\mathbb{R}^n$.

Theorem (Optimal nonlinear stability)

For every $u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{R}^n)$ there holds

$$\inf_{\phi \in Mob(\mathbb{S}^{n-1})} \fint_{\mathbb{S}^{n-1}} \left| \frac{1}{|V_n(u)|^{\frac{1}{n}}} \nabla_T u - \nabla_T \phi \right|^{n-1} \lesssim \mathcal{E}_{n-1}(u)\,.$$

◇ For $n = 3$: Luckhaus-Z. ('22, Invent. Math.) + linear stability $\forall n \geq 3$, leading to the nonlinear estimate in the $W^{1,\infty}$-vicinity of $Mob(\mathbb{S}^{n-1})$.

◇ For $n \geq 4$: Guerra-Lamy-Z. ('24).

◇ The result implies the estimate for $\mathbb{S}^{n-1}$-valued maps of degree ± 1, making it optimal in terms of scaling.

The (related) 2-dim H-functional

◇ Consider the functional $\mathcal{F} : \dot{H}^1(\mathbb{R}^2; \mathbb{R}^3) \to \mathbb{R}$ defined by

$$\mathcal{F}(u) := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \, \mathrm{d}\mathcal{L}^2 + \frac{2}{3} \int_{\mathbb{R}^2} \langle u, u_x \wedge u_y \rangle \, \mathrm{d}\mathcal{L}^2 .$$

◇ Relation to CMC-surfaces: The critical points $\omega \in \dot{H}^1(\mathbb{R}^2; \mathbb{R}^3)$ (*bubbles*);

$$\Delta\omega = 2\,\omega_x \wedge \omega_y \quad \text{in } \mathcal{D}'(\mathbb{R}^2) \, ,$$

are (branched) conformal parametrizations of unit spheres.

◇ Brezis-Coron ('84, ARMA) classified all such bubbles as

$$\omega(z) = \pi\left(\frac{P(z)}{Q(z)}\right) + b \, ,$$

where $P, Q \in \mathbb{C}[z]$, $b \in \mathbb{R}^3$, $\pi \colon \mathbb{C} \to \mathbb{S}^2$ is the inverse stereographic projection. If P/Q is irreducible and $k := \max\{\deg P, \deg Q\}$, then

$$\frac{1}{2} \int_{\mathbb{R}^2} |\nabla\omega|^2 \, \mathrm{d}\mathcal{L}^2 = 4\pi k \, .$$

They also proved a bubbling compactness result for Palais-Smale sequences.

Studying $\mathcal{F}$ near its critical points

- The linearized operator $\mathcal{F}''(\omega) : \dot{H}^1 \to \dot{H}^{-1}$ around a bubble ω is

$$\mathcal{F}''(\omega)[u] := -\Delta u + 2(u_x \wedge \omega_y + \omega_x \wedge u_y)\,.$$

- Since for some $k \in \mathbb{N}$ we can identify

$$\omega \in \mathcal{M}_k := \left\{(P, Q, b) \in \mathbb{C}[z] \times \mathbb{C}[z] \times \mathbb{R}^3 : \; P \text{ monic}\,, \; \max\{\deg P, \deg Q\} = k\right\},$$

infinitesimal variations tangent to M_k produce elements in $\ker \mathcal{F}''(\omega)$, so that

$$\dim \ker \mathcal{F}''(\omega) \geq 4k + 5\,.$$

- A bubble ω is non-degenerate, if all elements in $\ker \mathcal{F}''(\omega)$ arise in this way.

- Isobe ('91, Adv. Diff. Eq.), Chanillo-Malchiodi ('05, Comm. Anal. Geom.): Bubbles of degree 1 are non-degenerate.

- Sire-Wei-Zheng-Zhou ('23): The standard k-bubble, $k \geq 2$, corresponding to $P(z) = z^k$ and $Q(z) = 1$, is non-degenerate as well.

- Conjecture/Guess in these works: **All bubbles are non-degenerate!**

Theorem (Guerra-Lamy-Z., '24): This is not always the case!

Let $\omega \colon \mathbb{S}^2 \to \mathbb{R}^3$ be a bubble whose set of branch points is

$$\{|\nabla \omega| = 0\} =: \{p_1, \dots, p_n\}.$$

Then ω is degenerate iff $\exists$ a non-zero polynomial $R \in \mathbb{C}[z]$ with $\deg R \leq n - 4$:

$$h(z) := \frac{R(z)}{(z - p_1)\dots(z - p_n)}\,, \quad \mathrm{Res}_{p_j}\left(\frac{h}{(P/Q)'}\right) = 0 \quad \text{for } j \in \{1, \dots, n\}\,.$$

- The result is based on the characterization of extra eigenfunctions to

  $$\Delta f + |\nabla \omega|^2 f = 0\,,$$

  Montiel-Ros ('90, Conf. Proc. Berlin), Ejiri-Kotani ('93, Tokyo J. Math.) .

- Every degenerate bubble needs to have at least 4 branch points!

- Every bubble of degree $k \leq 2$ is non-degenerate!

- For $k = 3$, the only degenerate bubble is (up to a Möbius transformation)

  $$P(z) = z^3 + 2\,, \quad Q(z) = z\,.$$

Proof of nonlinear stability from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$, $n = 3$

- By a **contradiction/compactness argument** it suffices to prove the $W^{1,2}$-**local version** of the theorem, i.e., prove it for maps with

$$(i) \qquad \fint_{\mathbb{S}^2} u = 0\,, \quad \fint_{\mathbb{S}^2} \langle u, x\rangle = 1\,,$$

$$(ii) \quad \mathcal{E}_2(u) \ll 1\,,$$

$$(iii) \quad \|\nabla_T u - P_T\|_{L^2(\mathbb{S}^2)} \ll 1\,.$$

- For such maps, setting $w := u - \mathrm{id}$ and expanding the deficit, we get

$$\mathcal{E}_2(u) = Q_3(w) + o\left(\fint_{\mathbb{S}^2} |\nabla_T w|^2\right)\,.$$

- For $n \geq 4$, if u is $W^{1,\infty}$−close to id, we get

$$\mathcal{E}_{n-1}(u) = Q_n(w) + \mathcal{O}\left(\fint_{\mathbb{S}^{n-1}} |\nabla_T w|^3\right)\,.$$

Linear stability in the conformal case, $n \geq 3$

$$Q_n(w) := \frac{n}{2(n-1)} \fint_{\mathbb{S}^{n-1}} \left(|\nabla_T w|^2 + \frac{n-3}{n-1} (\mathrm{div}_{\mathbb{S}^{n-1}} w)^2 \right) - \frac{n}{2} \fint_{\mathbb{S}^{n-1}} \langle w, A(w) \rangle ,$$

where

$$A(w) := (\mathrm{div}_{\mathbb{S}^{n-1}} w) x - \sum_{j=1}^{n} x_j \nabla_T w^j ,$$

considered in the space

$$H_n := \left\{ w \in W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}^n) : \fint_{\mathbb{S}^{n-1}} w = 0 , \fint_{\mathbb{S}^{n-1}} \langle w, x \rangle = 0 \right\} .$$

Theorem (Luckhaus-Z., Linear stability, $n \geq 3$)

There exists $C_n > 0$ such that $\forall w \in H_n$,

$$Q_n(w) \geq C_n \fint_{\mathbb{S}^{n-1}} \left| \nabla_T w - \nabla_T (\Pi_{n,0} w) \right|^2 ,$$

where $\Pi_{n,0} : H_n \to H_{n,0}$ is the $W^{1,2}$-orthogonal projection on the kernel $H_{n,0} \cong \mathfrak{mob}(n-1)$ of Q_n in H_n (of dimension $n(n+1)/2$).

◇ When $\mathbf{n} = \mathbf{3}$, the optimal constant can be calculated explicitely.

◇ **Linear stability** $\Longrightarrow$ **Nonlinear stability**, via an application of an **Inverse Function Theorem** and a **topological argument** allowing us to find

$$\phi \in Mob_+(\mathbb{S}^2) \ \text{ s.t. } \ \Pi_{3,0}(u \circ \phi) = 0\,.$$

◇ The linear estimate is based on the fine interplay between A and $-\Delta_{\mathbb{S}^{n-1}}$:

Lemma (A-eigenvalue decomposition of spherical harmonics)

(i) For $n \geq 3$, $k \geq 1$, let

$$H_{n,k} := \left\{ w \in H_n : \ -\Delta_{\mathbb{S}^{n-1}} w = \lambda_{n,k} w\,, \quad \lambda_{n,k} := k(k+n-2) \right\}.$$

The L^2 self-adjoint operator $A(w)$ leaves $H_{n,k,\mathrm{sol}}$, $H^{\perp}_{n,k,\mathrm{sol}}$ invariant, where

$$H_{n,k,\mathrm{sol}} := \{ w \in H_{n,k} : \mathrm{div} w_h \equiv 0 \text{ in } B_1 \}\,.$$

(ii) For every $k \geq 1$,

$$H_{n,k,\mathrm{sol}} = H_{n,k,1} \oplus H_{n,k,2}\,, \quad H^{\perp}_{n,k,\mathrm{sol}} := H_{n,k,3}\,,$$

where $(H_{n,k,i})_{i=1}^{3}$ are the eigenspaces of A w.r.t. the eigenvalues $-k, 1, k+n-2$ respectively.

Sketch of proof, $n \geq 4$

◇ Reduce to u satisfying $(i) - (iii)$, $\Pi_{n,0}(u) = 0$, via the **qualitative version:**

Proposition (Strong compactness of minimizing sequences)

If $\mathcal{E}_{n-1}(u_j) \to 0$, then U.T.S. there exist $(\phi_j)_{j\in\mathbb{N}} \subset Mob(\mathbb{S}^{n-1})$ and $O \in \mathrm{O}(n)$ s.t.

$$\frac{u_j \circ \phi_j - \fint_{\mathbb{S}^{n-1}} u_j \circ \phi_j}{|V_n(u_j)|^{1/n}} \to O\mathrm{id} \ \text{ strongly in } W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{R}^n).$$

◇ $n = 3$: Brezis-Coron ('84, ARMA), Caldiroli-Musina ('06, ARMA).

◇ $n \geq 4$: Passing to $W^{1,n-1}$-equibounded Palais-Smale sequences, i.e.,

$$\Delta_{n-1} u_j + H_{u_j} J(u_j) \to 0 \ in \ (W^{1,n-1})^*,$$

which are strongly compact in $W^{1,q}$ $\forall q \in [1, n-1)$,

$$D_{n-1}(u_j) = D_{n-1}(u) + D_{n-1}(u_j - u) + o(1) \quad \text{(Brezis-Lieb lemma)}$$

$$V_n(u_j) = V_n(u) + V_n(u_j - u) + o(1) \quad \text{(weak convergence of minors)}.$$

- ◇ If $\mathcal{E}(u_j) \to 0$ is P. S. and $u_j \rightharpoonup u$ weakly in $W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{R}^n)$, where u is **non-constant**, by minimality and the isoperimetric inequality,

$$\begin{aligned} D_{n-1}(u) &= D_{n-1}(u_j) - D_{n-1}(u_j - u) + o(1) \\ &= |V_n(u) + V_n(u_j - u)|^{\frac{n-1}{n}} - D_{n-1}(u_j - u) + o(1) \\ &\leq |V_n(u)|^{\frac{n-1}{n}} + |V_n(u_j - u)|^{\frac{n-1}{n}} - D_{n-1}(u_j - u) + o(1) \\ &\leq D_{n-1}(u) + o(1)\,, \end{aligned}$$

so that **either**

$$V_n(u) = 0 \implies D_{n-1}(u) = 0 \quad \text{(contradiction)!}\,,$$

or

$$V_n(u_j - u) \to 0 \overset{\cdots}{\implies} D_{n-1}(u_j - u) \to 0\,,$$

i.e., the weak convergence in $W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{R}^n)$ is improved to strong!

- ◇ Apply a **concentration compactness argument** (P. L. Lions, '84 AIHPC). Use the Möbius-invariance to equi-spread the energy after precompositions, so that the resulting weak limit is non-constant and at the end a rigid motion.

◇ Prove the **local-$W^{1,n-1}$ nonlinear estimate**, i.e.,

Proposition

For every $w \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{R}^n)$ with

$$\fint_{\mathbb{S}^{n-1}} w = 0\,, \quad \fint_{\mathbb{S}^{n-1}} \langle w, x\rangle = 0\,, \quad \Pi_{n,0}(w) = 0\,,$$

$$\mathcal{E}_{n-1}(\mathrm{id} + w) \ll 1\,, \quad \|\nabla_T w\|_{L^{n-1}} \ll 1\,,$$

there holds

$$\mathcal{E}_{n-1}(\mathrm{id} + w) \gtrsim \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^{n-1}\,.$$

Basic ingredients for the proof

◇ **Expansion of** V_n controlling the error terms via Sobolev embedding, Hölder's and the parametric conformal-isoperimetric inequality.

◇ **A suitable lower Taylor-type inequality** instead of an expansion for D_{n-1}.

◇ A contradiction/compactness argument based on the corresponding **linear stability estimate**.

◇ After these reductions,

$$2\mathcal{E}_{n-1}(\mathrm{id}+w) \geq [D_{n-1}(\mathrm{id}+w)]^{\frac{n}{n-1}} - V_n(\mathrm{id}+w)\,.$$

◇ For V_n (which has polynomial structure):

$$V_n(\mathrm{id}+w) - \left(1+\frac{n}{2}\fint_{\mathbb{S}^{n-1}}\langle w, A(w)\rangle\right) = \sum_{k=2}^{n-2}\fint_{\mathbb{S}^{n-1}}\langle w, P_k(\nabla_T w)\rangle + V_n(w)$$
$$\lesssim \left(\fint_{\mathbb{S}^{n-1}}|\nabla_T w|^2\right)^{1+\alpha} + \left(\fint_{\mathbb{S}^{n-1}}|\nabla_T w|^{n-1}\right)^{1+\beta}$$

◇ For D_{n-1} (which does not have polynomial structure), the key tool is

Algebraic lemma (Figalli-Zhang, '22, Duke Math. J.)

Let $p \geq 2$ and $X, Y \in \mathbb{R}^m$. For every $\kappa > 0$ there exists $c_\kappa > 0$ s.t.

$$|X+Y|^p - \left(|X|^p + p|X|^{p-2}\langle X, Y\rangle\right) \geq \frac{1-\kappa}{2}\left(p|X|^{p-2}|Y|^2 + |W|^{p-2}(|X|-|X+Y|)^2\right)$$
$$+ c_\kappa |Y|^p\,,$$

for an appropriate weight $W := W(X, X+Y)$.

- Applying it for $m := n(n-1)$, $p := n-1$, $X := P_T$, $Y := \nabla_T w$, ...

$$2\mathcal{E}_{n-1}(\mathrm{id} + w) \geq c_\kappa \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^{n-1} + (1-\kappa)\tilde{Q}_n(w) - \frac{\kappa n}{2} \fint_{\mathbb{S}^{n-1}} \langle w, A(w) \rangle - c_n \Big(\fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 \Big)^{1+\alpha},$$

$\tilde{Q}_n(w) := Q_n(w) + R_n(\nabla_T w)$ (the remainder coming from the weighted interpolant).

- Show that the red term is non-negative, via the following:

Lemma (Mildly-nonlinear stability)

$\forall\ C, \alpha > 0$, $|c| \ll 1$, $\exists\ 0 < \theta \ll 1$ s.t. if w as above has $\fint_{\mathbb{S}^{n-1}} |\nabla_T w|^{n-1} \leq \theta$,

$$\tilde{Q}_n(w) \geq c \fint_{\mathbb{S}^{n-1}} \langle w, A(w) \rangle + C \Big(\fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 \Big)^{1+\alpha}.$$

- If not, along a ($W^{1,2}$-rescaled) contradicting sequence, we obtain a weak limit $\hat{w} \in H_n$, with $\Pi_{n,0}(\hat{w}) = 0$, for which ...

$$Q_n(\hat{w}) \leq |c| \fint_{\mathbb{S}^{n-1}} |\nabla_T \hat{w}|^2 \quad \text{(contradiction) ! ! !}$$

Ευχαριστώ πολύ !!!