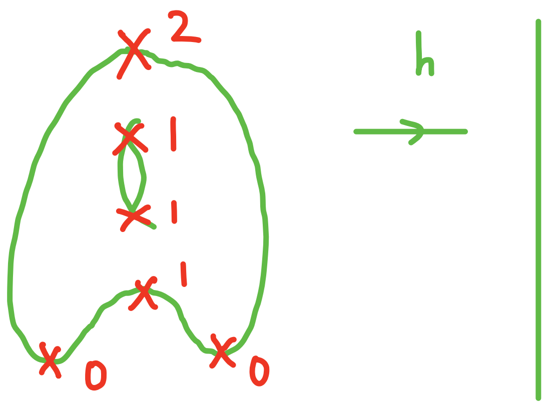
Witten instanton complexes on stratified pseudomanifolds

Gayana Jayasinghe

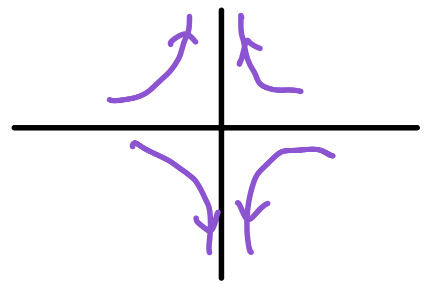
Outline:

- Lefschitz $\longrightarrow$ Morse
  de Rham
  Twisted Dolbeault
- Instanton complexes,
  Morse inequalities
- Applications — Maths
  Physics

∃ de Rham Lefschetz fixed point theorem



$(-1)^{\#[\text{expanding directions}]}$



$$(-1)^2 + 3(-1)^1 + 2(-1)^0 = \sum_{k=0}^{n} (-1)^k \dim \mathcal{H}^k(X)$$

$$1 - 3 + 2$$

∃ de Rham Morse inequalities

$$b^2 + 3b^1 + 2b^0 = b^2 + 2b^1 + b^0 + (1+b)$$

$$\sum_{a \in \mathrm{Crit}(h)} \sum_{k=0}^{n} b^k \dim \mathcal{H}^k_B(\mathcal{U}_a) = \sum_{k=0}^{n} b^k \dim \mathcal{H}^k(X) + (1+b) \sum_{k=0}^{n-1} b^k Q_k$$

∃ de Rham Morse inequalities

Thm: Let X be a smooth manifold with a Morse function h. Then

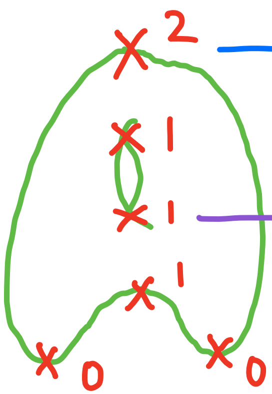
$$\sum_{a \in \mathrm{Crit}(h)} \sum_{k=0}^{n} b^k \dim \mathcal{H}_B^k(\mathcal{U}_a) = \sum_{k=0}^{n} b^k \dim \mathcal{H}^k(X) + (1+b) \sum_{k=0}^{n-1} b^k Q_k$$

where Q_k non-negative integers.

$$\mathcal{U}_a = \underbrace{\mathcal{U}_{a,1}}_{\text{attracting}} \times \underbrace{\mathcal{U}_{a,2}}_{\text{expanding}}$$

$$\mathcal{H}_B^k(\mathcal{U}_a) = \bigoplus_{k=k_1+k_2} \mathcal{H}^{k_1}(\mathcal{U}_{a,1}) \otimes \mathcal{H}^{k_2}(\mathcal{U}_{a,2}, \partial \mathcal{U}_{a,2})$$

$$\mathcal{H}^k_B(\mathcal{U}_a) = \bigoplus_{k=k_1+k_2} \mathcal{H}^{k_1}(\mathcal{U}_{a,1}) \otimes \mathcal{H}^{k_2}(\mathcal{U}_{a,2}, \partial \mathcal{U}_{a,2})$$



$$pt \times \mathbb{D}^2$$

$$1 \wedge \mathrm{dvol}^2 \in \mathcal{H}^2_B(\mathcal{U}_a)$$

$$\mathbb{D}^1 \times \mathbb{D}^1$$

$$1 \wedge \mathrm{dvol}^1 \in \mathcal{H}^1_B(\mathcal{U}_a)$$

$$\mathbb{D}^2 \times pt$$

$$1 \wedge 1 \quad \in \mathcal{H}^0_B(\mathcal{U}_a)$$

$$b^2 + 3b^1 + 2b^0$$

$$= \sum_{a \in \mathrm{Crit}(h)} \sum_{k=0}^{n} b^k \dim \underbrace{\mathcal{H}^k_B(\mathcal{U}_a)}_{\text{Categorifies Morse polynomial}}$$

Thm: (Witten - de Rham instanton complex)

Let X be a smooth manifold with a Morse function h. Then $\exists$ finite dim complex

$$0 \longrightarrow F_0 \xrightarrow{d_\varepsilon} F_1 \xrightarrow{d_\varepsilon} F_2 \cdots\cdots F_{n-1} \xrightarrow{d_\varepsilon} F_n \longrightarrow 0$$

i) quasi-isomorphic to the de Rham complex,

ii) with $F_k = \bigoplus_{a \in \mathrm{Crit}(h)} \bigoplus_{k=0}^{n} \dim \mathcal{H}^k_B(\mathcal{U}_a)$.

(Witten, Helfer-Sjöstrand, ------)

Corollary: "Lefschetz-Morse" inequalities

Q: Given a Lefschetz fixed point theorem

i] Are there Morse inequalities generalizing it?

ii] Are there Categorifications of the Morse polynomial?

Witten formulated versions for

Twisted Lefschetz - Riemann Roch / Atiyah - Bott Lefschetz

on smooth manifolds

Holomorphic Lefschetz fixed point theorem (Atiyah-Bott)

Given a smooth complex manifold X, $\dim_{\mathbb{R}}(X) = 2n$,

a Holomorphic bundle E,

we have the (twisted) Dolbeault complex $\mathcal{D}(X) := (\Omega^{0,q}(X;E), \bar{\partial}_E)$

$$0 \longrightarrow \Omega^{0,0}(X;E) \xrightarrow{\bar{\partial}_E} \Omega^{0,1}(X;E) \xrightarrow{\bar{\partial}_E} \cdots \Omega^{0,n}(X;E) \xrightarrow{0} 0$$

Consider a holomorphic self map $f: X \longrightarrow X$, which lifts to E,

global holomorphic Lefschetz number

$L(X, \mathcal{D}(X), f)$

$$:= \sum_{q=0}^{n} (-1)^q \operatorname{tr}\left(f^*\big|_{H^q(\underbrace{\mathcal{D}(X)}_{})}\right) = \sum_{a \in \operatorname{fix}(f)} \frac{\operatorname{Tr}\ f_p^*: E|_p \to E|_p}{\det\left(\operatorname{Id} - df'\big|_a\right)}$$

Cohomology of the complex

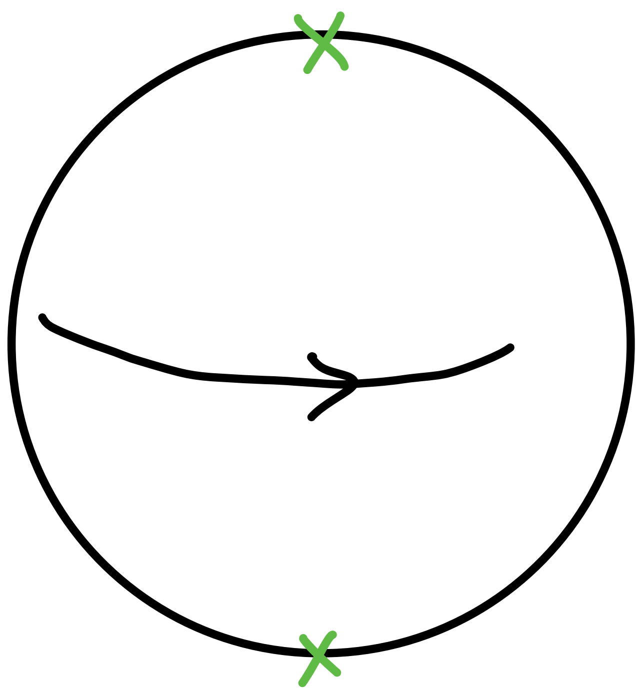
Global Dolbeault cohomology

for simple isolated f.p. a

df', holomorphic differential on $T^{1,0}X$

linear algebra on tangent space

Kähler Hamiltonian (Morse) function



$\mathbb{CP}^1 = S^2$

$h = \cos\phi$

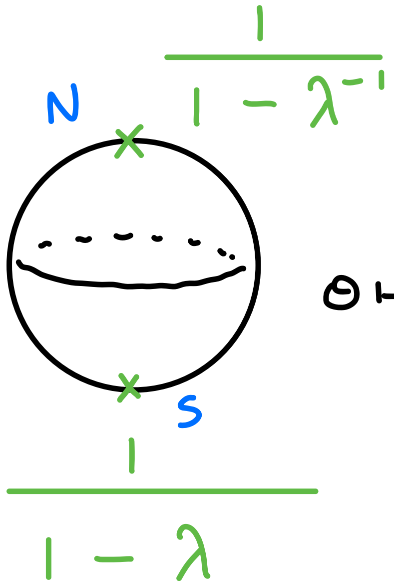
$\omega = \sin\phi \, d\theta \wedge d\phi$

$i_{\partial_\theta}\omega = \sin\phi \, d\phi = dh$

$f_\alpha(\theta, \phi) = (\theta + \alpha, \phi)$

$f_\alpha^* h = h$

Witten :



$\theta \longmapsto \theta + \alpha$

$$\frac{1}{1-\lambda} + \frac{1}{1-\lambda^{-1}} = 1$$

$$\frac{1}{1-\lambda} + \frac{(-1)\,\lambda}{1-\lambda} = 1$$

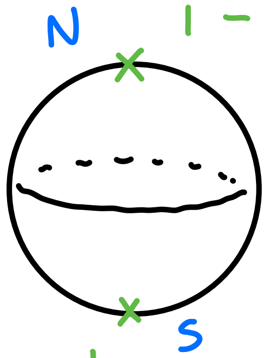
Trace formula??

$$\sum_{k=0}^{\infty} \lambda^k + (-1)^1 \sum_{k=1}^{\infty} \lambda^k = 1$$

$$\sum_{k=0}^{\infty} \lambda^k + (b)^1 \sum_{k=1}^{\infty} \lambda^k = 1 + (1+b) \sum_{k=1}^{\infty} \lambda^k$$

Witten:

$$\frac{1}{1-\lambda^{-1}}$$



$$\frac{1}{1-\lambda}$$

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$$\frac{(-1)\,\lambda^{-1}}{1-\lambda^{-1}} + \frac{1}{1-\lambda^{-1}} = 1$$

Trace formula??

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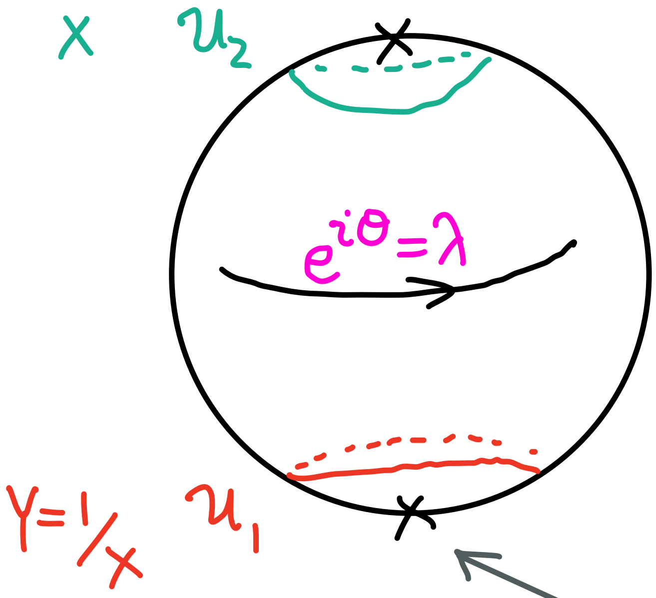
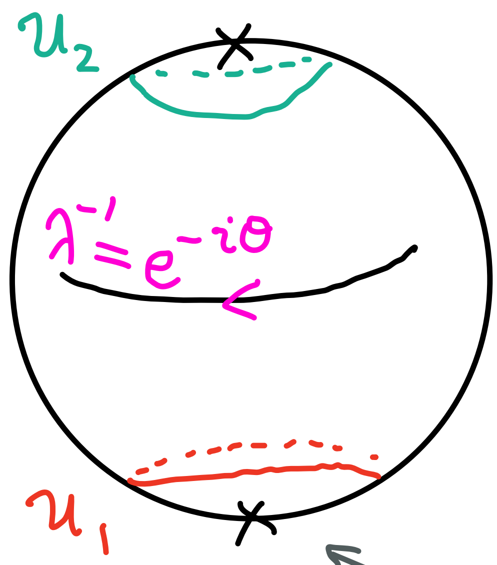
$$\sum_{k=0}^{\infty} \lambda^k + (b)^1 \sum_{k=1}^{\infty} \lambda^k = 1 + (1+b) \sum_{k=1}^{\infty} \lambda^k$$

$$(b') \sum_{k=1}^{\infty} \lambda^{-k} + \sum_{k=0}^{\infty} \lambda^{-k} = 1 + (1+b) \sum_{k=1}^{\infty} \lambda^{-k}$$





$$\begin{array}{ccc}
1 & & \\
x & \xleftrightarrow{\lambda^1} & d\bar{Y} \\
x^2 & \xleftrightarrow{\lambda^2} & \bar{Y}\,d\bar{Y} \\
x^3 & \xleftrightarrow{\lambda^3} & \bar{Y}^2\,d\bar{Y} \\
\vdots & & \vdots
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & & 1 \\
d\bar{X} & \xleftrightarrow{\lambda^{-1}} & Y \\
\bar{X}\,d\bar{X} & \xleftrightarrow{\lambda^{-2}} & Y^2 \\
\bar{X}^2\,d\bar{X} & \longleftrightarrow & Y^3
\end{array}$$

$$1 + (1+b)\sum_{k=1}^{\infty} \lambda^k \qquad\qquad 1 + (1+b)\sum_{k=1}^{\infty} \left(e^{-i\theta}\right)^k$$

Common polynomial $= 1$

] de Rham Morse inequalities

Let X be a smooth manifold with a Morse function h. Then

$$\sum_{a \in \mathrm{Crit}(h)} \sum_{k=0}^{n} b^k \dim \mathcal{H}^k_B(\mathcal{U}_a) = \sum_{k=0}^{n} b^k \dim \mathcal{H}^k(X) + (1+b) \sum_{k=0}^{n-1} b^k Q_k$$

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$$\frac{\ker \delta}{\operatorname{im} \delta} = \mathcal{H}^{k_2}_{\delta}(\mathcal{U}_{a,2})$$

Holomorphic Morse inequalities [Witten, Mathai-Wu]

Let X be a Kähler manifold with a Hamiltonian Morse function h. Then

$$\sum_{a\in \mathrm{Crit}(h)} \sum_{q=0}^{n} (b)^q \,\mathrm{Tr}\, T_\theta \Big|_{H_B^{0,q}(\mathcal{U}_a)} = \sum_{k=0}^{n} b^k \,\mathrm{Tr}\, T_\theta \Big|_{H^{0,q}(X)} + (1+b)\sum_{q=0}^{n-1} b^q Q_q$$

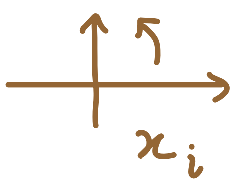
where Q_q are power series in $\lambda = e^{i\theta}$, with non-negative integer coefficients, T_θ induced by S^1_θ action.

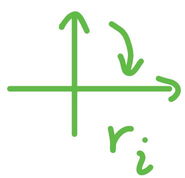
$$H_B^{0,q}(\mathcal{U}_a) = \bigoplus_{q_1+q_2=q} H^{0,q_1}_{\bar\partial}(\mathcal{U}_{a,1}) \otimes H^{0,q_2}_{\bar\partial^*}(\mathcal{U}_{a,2})$$

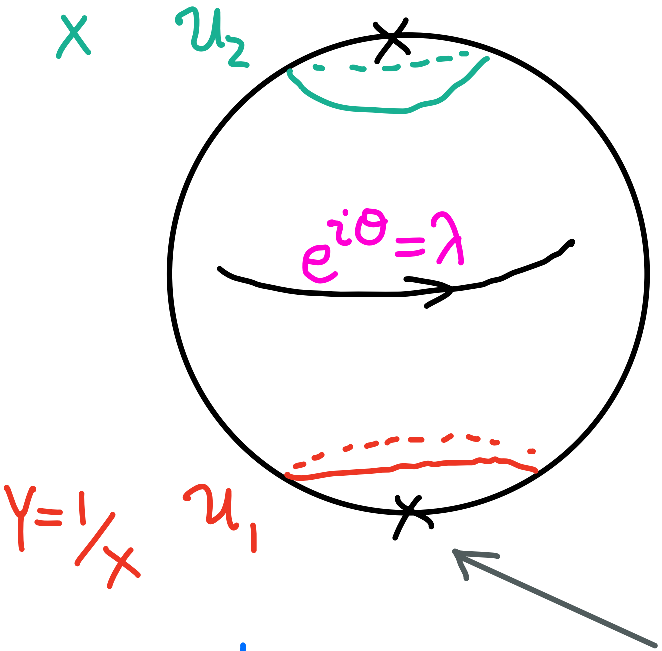
$$\mathcal{U}_a = \underbrace{\mathcal{U}_{a,1}}_{\text{anti-clockwise action}} \times \underbrace{\mathcal{U}_{a,2}}_{\text{clockwise action}}$$

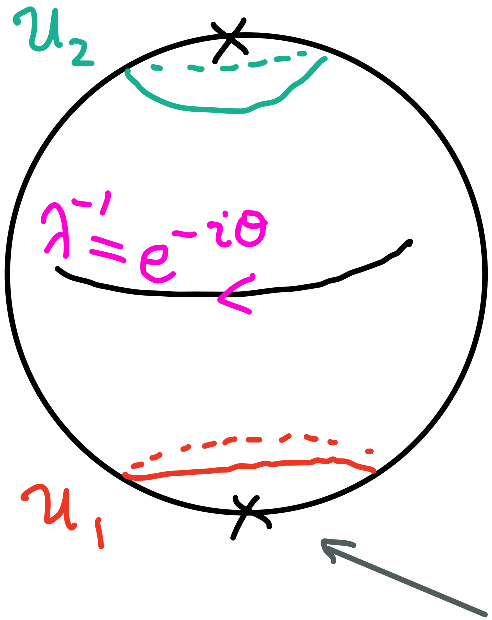
$$h = \sum \gamma_i x_i^2 - \sum \gamma_i r_i^2$$

$$dh = i_V \omega$$









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common polynomial $= 1$

] Holomorphic Morse inequalities [J.]

Let X be a stratified pseudomanifold with a Kähler wedge metric & a Kähler Hamiltonian Morse function h. Then

$$\sum_{a \in Crit(h)} \sum_{q=0}^{n} (b)^q \, Tr\, T_\theta \Big|_{H_B^{0,q}(\mathcal{U}_a)} = \sum_{k=0}^{n} b^k \, Tr\, T_\theta \Big|_{H^{0,q}(X)} + (1+b) \sum_{q=0}^{n-1} b^q Q_q$$

where Q_q are power series in $\underline{\lambda = e^{i\theta}}$, with non-negative integer coefficients, T_θ induced by S^1_θ action.

] $\exists$ a finite dim. instanton complex for each fixed eigenvalue μ of $i h_V$

[each power of $\underline{\lambda^\mu}$]

$$0 \longrightarrow F_0 \xrightarrow{\bar\partial_\varepsilon} F_1 \xrightarrow{\bar\partial_\varepsilon} F_2 - - - - F_{n-1} \xrightarrow{\bar\partial_\varepsilon} F_n \longrightarrow 0$$

i] Holomorphic Morse inequalities [J.]

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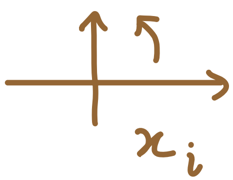
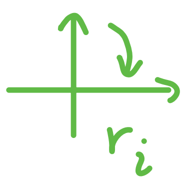
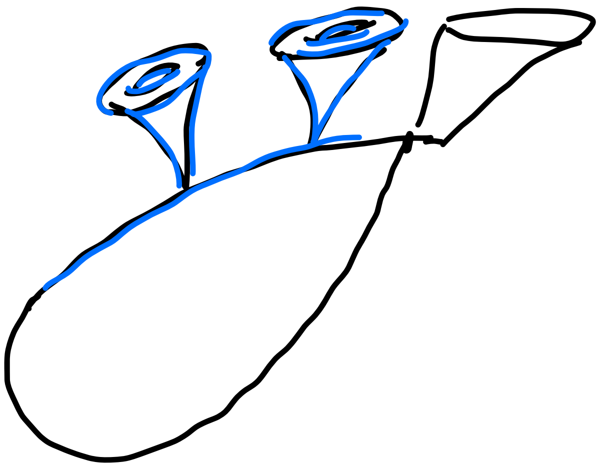


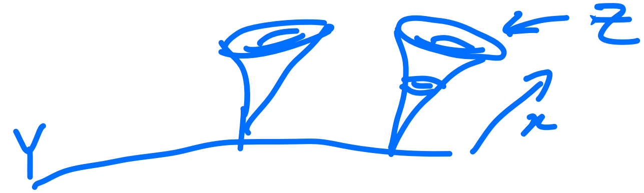


i] $(b=-1)$ 2023 (J.)

ii] instanton complex

Wedge metric:





$$\underbrace{dy^2 + dx^2 + x^2 g_Z} + \mathcal{O}(x^\delta)$$

conformally

$\delta > 1$

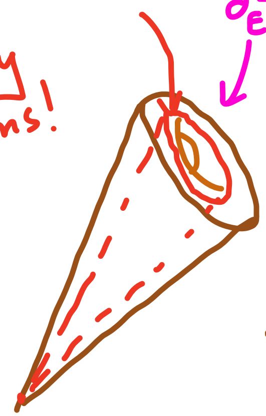
$$dx^2 + x^2\alpha^2 + x^2 g_{Z/F_\zeta}$$

$$\omega_1 = 2\,dx \wedge x\alpha - x^2\,d\alpha$$

$$\omega_c = (c+1)\,dx \wedge x^c\alpha - x^{c+1}\,d\alpha$$

ideal boundary conditions!

$\bar{\partial}_E$-Neumann / $\bar{\partial}_E^*$-Neumann boundary conditions



$C_x(Z)$

$$i_\zeta\, \omega_1 = 2x\,dx = dx^2$$

$$C_{x_1}(Z_1) \times C_{x_2}(Z_2) \times \cdots$$

$$h \simeq \sum_i \gamma_i x_i^2$$

$$\ker(\bar{\partial}_E + \bar{\partial}_E^*) = \frac{\ker \bar{\partial}_E}{\operatorname{im} \bar{\partial}_E}$$

$$g \simeq dy^2 + dx^2 + x^2 g_Z + \mathcal{O}(x^\delta)$$

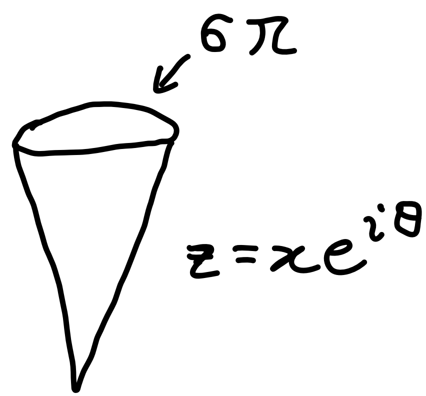
$$\mathcal{D}_\alpha(P_F) = \{ u \in L^2(X;F) \mid P_F u \in x^\alpha L^2(X;F) \}$$

$$\alpha = 0^+ \longrightarrow \max$$

$$\alpha = 1^- \longrightarrow \min$$

$$\exists\ \mathcal{D}_{\min}(\bar{\partial}_E) \cap \mathcal{D}_{\max}(\bar{\partial}_E^*) \longrightarrow \text{“Topological !!”}$$

$$\exists\ \mathcal{D}_{1/2}(\bar{\partial}_E) \cap \mathcal{D}_{1/2}(\bar{\partial}_E^*) = \mathcal{D}_{VAPS}(\bar{\partial}_E + \bar{\partial}_E^*)$$



$$\mathcal{D}_{\min}(\bar{\partial}) \supseteq \{1, z^{1/3}, z^{2/3}, z^1, z^{4/3}, \ldots\}$$

$$\mathcal{D}_{1/2}(\bar{\partial}) \supseteq \{z^{-1/3}, 1, z^{1/3}, z^{2/3}, z^1, z^{4/3}, \ldots\}$$

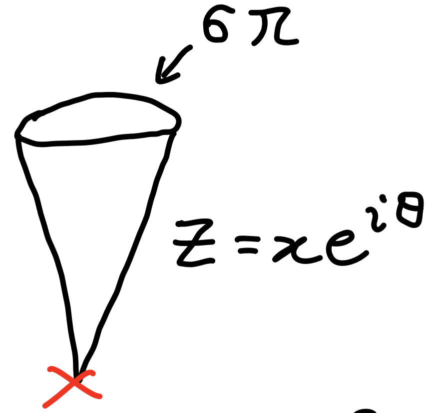
$$\mathcal{D}_{\max}(\bar{\partial}) \supseteq \{z^{-2/3}, z^{-1/3}, 1, z^{1/3}, z^{2/3}, z^1, z^{4/3}, \ldots\}$$

$$z = x e^{i\theta}$$

Serré dual complex:

$$\Omega^{0,n-q}(X;E^*\otimes K)=\Omega^{n,n-q}(X;E^*)$$

$$\mathcal{D}_{\alpha,SD}(\bar\partial_E)=\mathcal{D}_\alpha(\bar\partial_E^*) \qquad \bar\partial^*_{E^*}=-*\bar\partial_E*$$



Adjoint complex:

$$\left(L^2\Omega^{0,n-q}(X;E),\ \mathcal{D}_{1-\alpha}(\bar\partial^*),\ \bar\partial^*\right)$$

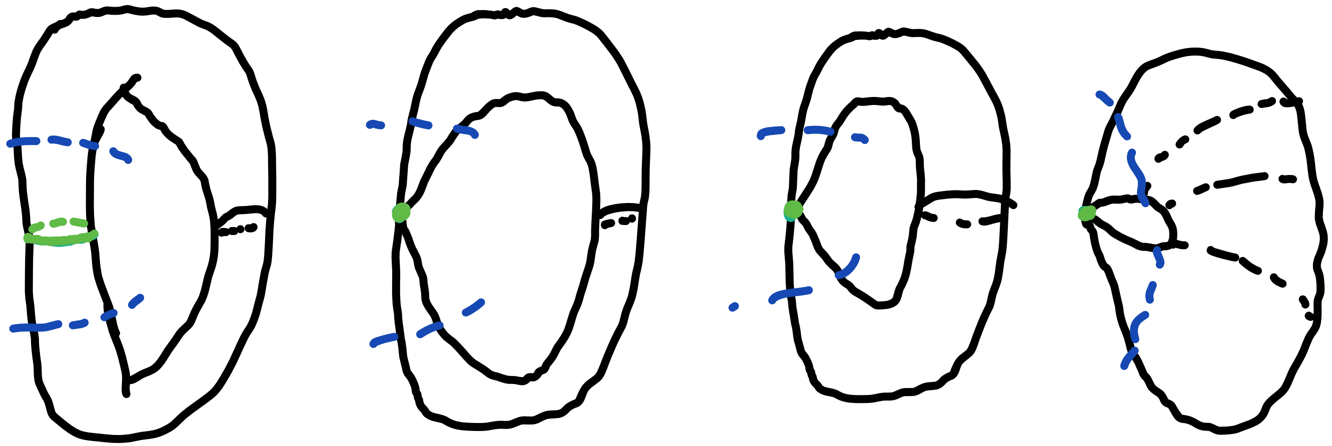
$$z=xe^{i\theta}$$

$\mathcal{D}_{min}(\bar\partial)\supseteq\{1, z^{1/3},\ldots$ $\qquad \mathcal{D}_{max}(\bar\partial^*)\supseteq\{d\bar z,\ \bar z^{1/3}d\bar z,\ \bar z^{2/3}d\bar z$

$\mathcal{D}_{1/2}(\bar\partial)\supseteq\{z^{-1/3}, 1, z^{1/3},\ldots$ $\qquad \mathcal{D}_{1/2}(\bar\partial^*)\supseteq\{\bar z^{1/3}d\bar z,\ \bar z^{2/3}d\bar z$

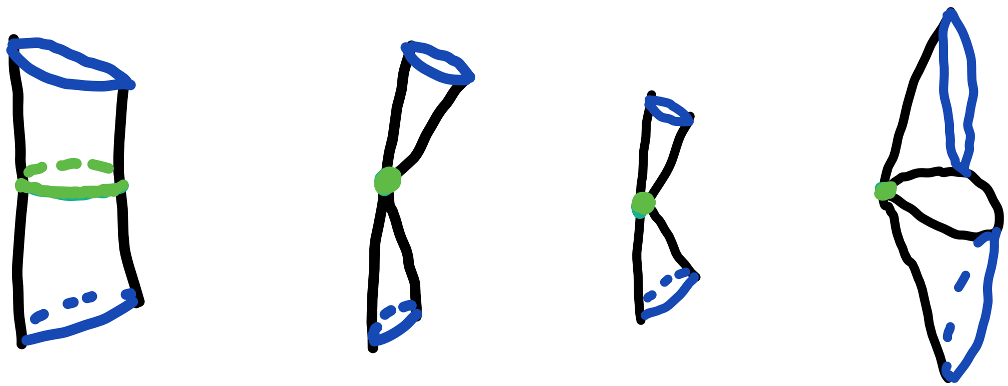
$\mathcal{D}_{max}(\bar\partial)\supseteq\{z^{-2/3}, z^{-1/3}, 1, z^{1/3},\ldots$ $\qquad \mathcal{D}_{min}(\bar\partial^*)\supseteq\{\bar z^{2/3}d\bar z$

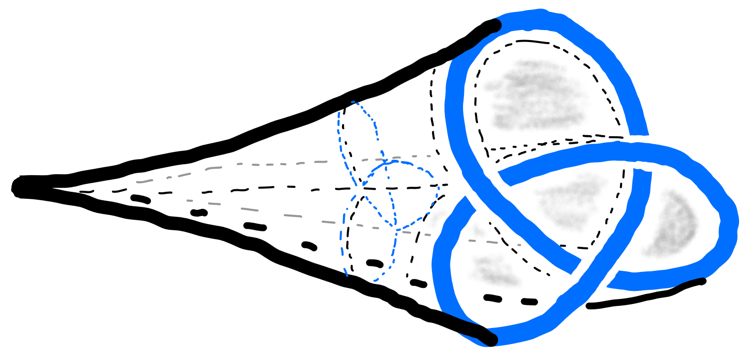
Example: $ZY^2 - X^3 = 0$ in $\mathbb{CP}^2_{[X:Y:Z]}$



$y^2 = x^3 + \underline{a}x + \underline{b}$

Wedge metric!





$b \longrightarrow 0$ $a \longrightarrow 0$

$y^2 = x^3$ in $\mathbb{R}^2$

in $\mathbb{C}^2$

Example: $ZY^2 - X^3 = 0$ in $\mathbb{CP}^2_{[X:Y:Z]}$

$\exists$ $(\lambda)[X:Y:Z] \longrightarrow [\lambda^2 X : \lambda^3 Y : Z]$

$\exists$ Two fixed points

singular $Z=1$, $X=0=Y$

smooth at $[0:1:0]$

$Y=1$, $Z=X^3$

$$\frac{1}{1-\lambda^{-1}}$$

Local ring $x = X/Z$, $y = Y/Z$

normalization map $t \longrightarrow (t^2, t^3) = (x, y)$

$t \longrightarrow \lambda t$

$\exists$ BFQ $\overline{\mathbb{C}[x,y]/(y^2-x^3)} = \{1, \cancel{t}, t^2, t^3, \ldots\}$

$$(1+\lambda^2+\lambda^3+\ldots) + \frac{1}{1-\lambda^{-1}} = 1-\lambda$$

Example: $ZY^2 - X^3 = 0$ in $\mathbb{CP}^2_{[X:Y:Z]}$

$$\exists\ (\lambda)[X:Y:Z] \longrightarrow [\lambda^2 X : \lambda^3 Y : Z]$$

$\exists$ Two fixed points

smooth at $[0:1:0]$

$Y=1, \quad Z = X^3$

$$\frac{b\lambda}{1-\lambda}$$

singular $Z=1$, $X=0=Y$ $\left(x = X/Z,\ y = Y/Z\right)$

normalization map $t \longrightarrow (t^2, t^3) = (x,y)$

$$t \longrightarrow \lambda t$$

Local ring

$t = y/x$

$\exists$ BFQ $\quad \overline{\mathbb{C}[x,y]/(y^2-x^3)} = \{1, \cancel{t}, t^2, t^3, \ldots\}$

$$(1 + \lambda^2 + \lambda^3 + \ldots) + \frac{\lambda^{-1}}{1-\lambda^{-1}} = 1 - \underline{\lambda}$$

$h^{1,1}$

$h^{1,0}$ $\quad$ $\overline{\left(\frac{dx}{2y}\right)}$

1

Poincaré residue

$$\frac{dy\wedge dx}{y^2 - x^3} = \frac{dx}{2y} \rightarrow \frac{\lambda^2}{\lambda^3} = \frac{1}{\lambda}$$

$$\overline{\left(\frac{dx}{2y}\right)} \Big\downarrow \quad \left(1/\lambda^*\right) \xrightarrow[=]{\text{if } s'c\ \mathbb{C}^*} \lambda^{-1}$$

geometric endo.

$t = y/x$

1] BFQ $\overline{\mathbb{C}[x,y]/(y^2-x^3)} = \{1, \cancel{t}, t^2, t^3, \ldots\}$

$(1+\cancel{\lambda}+\lambda^2+\lambda^3+\ldots) + b(\lambda+\lambda^2+\lambda^3+\ldots) = 1 + \underline{b\lambda} + (1+b)[\lambda^2+\lambda^3+\ldots]$

2] $D_{min}(\bar\partial)$ $\{1, t, t^2, t^3, \ldots\}$ [J. $\alpha = 1$]

$(1+\lambda+\lambda^2+\lambda^3+\ldots) + b(\lambda+\lambda^2+\lambda^3+\ldots) = 1 + (1+b)[\lambda+\lambda^2+\lambda^3+\ldots]$

3] $D_{max}(\bar\partial)$ $\{t^{-1} = x/y, 1, t, t^2, \ldots\}$ [J. $\alpha = 0$]

$(\lambda^{-1}+1+\lambda+\lambda^2+\lambda^3+\ldots) + b(\lambda+\lambda^2+\lambda^3+\ldots) = \lambda^{-1}+1 + (1+b)[\lambda+\lambda^2+\lambda^3+\ldots]$

4] Lott $\{1, t, t^2, t^3, \ldots\}^0 \oplus \{t\}^1$

$(1+\lambda+\lambda^2+\lambda^3+\ldots) + b(\lambda+\lambda+\lambda^2+\lambda^3+\ldots) = 1 + b\lambda + (1+b)[\lambda+\lambda^2+\lambda^3+\ldots]$

1,4] same L.R.R., different Morse (local)

$\because$ iso in K-theory

$\exists$ **Rigidity** $\iff$ $\mathrm{Tr}\, T_\theta\, \mathcal{H}^k(\mathcal{P}(X)) = \mathrm{Tr}\, \underbrace{T_{\theta=0}}_{\mathrm{Id}}\, \mathcal{H}^k(\mathcal{P}(X))$

$\exists$ **Thm.** (J.)

i) L^2- de-Rham instanton complex & Dolbeault inst. comp. $\mathcal{D}_{min}(\bar\partial)$ are rigid!

ii) Rigid subcomplex of $\mathcal{D}_\alpha(\bar\partial)$ instanton complex for $\alpha \in [0,1]$ $\cong L^2$ de Rham complex

(Pf hinges on Sasaki geometry.)

For $k < \dim(Z)/2$

$$\mathcal{H}^k(Z) = \mathcal{H}^k_{\mathcal{F}_{Reeb}}(Z)$$

$Z^2 - XY = 0$ in $\mathbb{CP}^3$ <u>normal</u>, <u>A_1</u>

$(\lambda, \mu)\ [W; X; Y; Z] \longrightarrow [W; \lambda^2 X; \mu^2 Y; \lambda\mu Z]$

Fixed points $\underbrace{[1;0;0;0]}_{\text{Singular}}$

$$\mathcal{H}^q(\mathcal{P}(\mathcal{U}_0)) = \begin{cases} 0 & q>0 \\ \mathcal{O}(\mathcal{U}_0) & q=0 \end{cases}$$

$W=1$, $x = \frac{X}{W}$ $\quad y = \frac{Y}{W}$ $\quad z = \frac{Z}{W}$

$\mathcal{O}(\mathcal{U}_0) = \underset{\sim}{R}[1, z]$

Ring generated by x & y } local ring

$x \longrightarrow \lambda^2 x \quad y \longrightarrow \mu^2 y \quad z \longrightarrow \lambda\mu z$

$$\mathrm{tr}_{re}\, f^*(\mathcal{O}(\mathcal{U}_0)) = \frac{1+\lambda\mu}{(1-\lambda^2)(1-\mu^2)}$$

$Z^2 - XY = 0$ in $\mathbb{CP}^3$ $\underline{\text{normal}}$, $\underline{A_1}$

$(\lambda,\mu)\,[W;X;Y;Z] \longrightarrow [W;\lambda^2 X;\mu^2 Y;\lambda\mu Z]$

Fixed points $\underbrace{[1;0;0;0]}_{\text{singular}}$ $\qquad$ $\underbrace{[0;1;0;0],\ [0;0;1;0]}_{\text{smooth}}$

$$\mathcal{H}^q(\mathcal{P}(\mathcal{U}_0)) = \begin{cases} 0 & q>0 \\ \mathcal{O}(\mathcal{U}_0) & q=0 \end{cases}$$

$W=1$, $x = \frac{X}{W}$ $\quad y = \frac{Y}{W}$ $\quad z = \frac{Z}{W}$

$$\mathcal{O}(\mathcal{U}_0) = \underbrace{R}_{\text{Ring generated by } x \,\&\, y}[1,z]$$

Ring generated by x & y } local ring

$x \longrightarrow \lambda^2 x \quad y \longrightarrow \mu^2 y \quad z \longrightarrow \lambda\mu z$

$$\mathrm{tr}_{re}\, f^*(\mathcal{O}(\mathcal{U}_0)) = \frac{1+\lambda\mu}{(1-\lambda^2)(1-\mu^2)}$$

$z^2 - y = 0$

$\mathcal{O}(\mathcal{U}_1) = \{z, w\}$

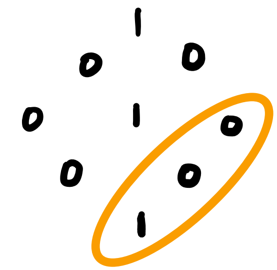
$z = \frac{Z}{X} \longrightarrow \frac{\lambda\mu}{\lambda^2} z$

$w \longrightarrow \frac{1}{\lambda^2} w$

$$\frac{1}{(1-\mu/\lambda)(1-1/\lambda^2)}$$

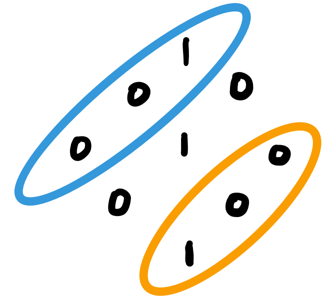
$$\frac{1}{(1-\lambda/\mu)(1-1/\mu^2)}$$

$$\Big] \; \frac{1+\lambda\mu}{(1-\lambda^2)(1-\mu^2)} + \frac{1}{(1-\mu/\lambda)(1-1/\lambda^2)} + \frac{1}{(1-\lambda/\mu)(1-1/\mu^2)} = \frac{1}{}$$



E = trivial bundle

1] $$\frac{1+\lambda\mu}{(1-\lambda^2)(1-\mu^2)} + \frac{1}{(1-\mu/\lambda)(1-1/\lambda^2)} + \frac{1}{(1-\lambda/\mu)(1-1/\mu^2)} = \underline{1}$$



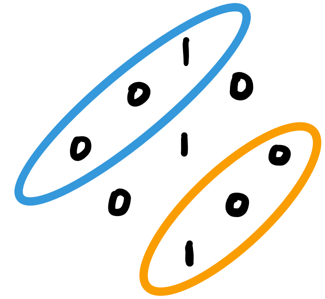
$$E = \Lambda^{2,0}(X) = K_X$$

E = trivial bundle

2] $$\frac{1+\lambda\mu\,(\lambda\mu)}{(1-\lambda^2)(1-\mu^2)} + \frac{\mu/\lambda \cdot 1/\lambda^2}{(1-\mu/\lambda)(1-1/\lambda^2)} + \frac{\lambda/\mu \cdot 1/\mu^2}{(1-\lambda/\mu)(1-1/\mu^2)} = \underline{1}$$

$$\Omega^{0,q}(X; K_X) = \Omega^{n,q}(X; \mathbb{C})$$

1] $\dfrac{1+\lambda\mu}{(1-\lambda^2)(1-\mu^2)} + \dfrac{1}{(1-\mu/\lambda)(1-1/\lambda^2)} + \dfrac{1}{(1-\lambda/\mu)(1-1/\mu^2)} = \underline{1}$



$E = \Lambda^{2,0}(X) = K_X$

$E =$ trivial bundle

2] $\dfrac{1+\lambda\mu\ (\lambda\mu)}{(1-\lambda^2)(1-\mu^2)} + \dfrac{\mu/\lambda \cdot 1/\lambda^2}{(1-\mu/\lambda)(1-1/\lambda^2)} + \dfrac{\lambda/\mu \cdot 1/\mu^2}{(1-\lambda/\mu)(1-1/\mu^2)} = \underline{1}$

$\Omega^{0,q}(X; K_X) = \Omega^{n,q}(X;\mathbb{C})$

$\lambda\mu = \mathrm{tr}\, f^* \Big| \underbrace{\mathrm{Res}_{\mathbb{C}^3|X} \dfrac{dx\wedge dy \wedge dz}{z^2-xy}}$

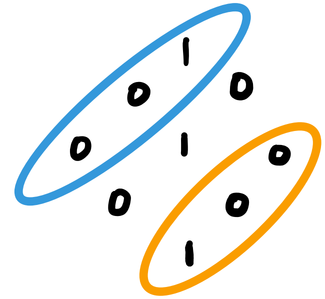
Poincaré Residue

$\mathrm{tr}\, f^*(z\cdot w) = \mu/\lambda \cdot 1/\lambda^2$

$\left[\dfrac{dx\wedge dy}{2z}\right] \longrightarrow \dfrac{\lambda^2\mu^2}{\lambda\mu} = (\lambda\mu)$

$\in L^2$

$$1]\quad \frac{1+\lambda\mu}{(1-\lambda^2)(1-\mu^2)} + \frac{1}{(1-\mu/\lambda)(1-1/\lambda^2)} + \frac{1}{(1-\lambda/\mu)(1-1/\mu^2)} = \underline{1}$$



$$E = \Lambda^{2,0}(X) = K_X$$

$E=$ trivial bundle

$$2]\quad \frac{1+\lambda\mu\,(\lambda\mu)}{(1-\lambda^2)(1-\mu^2)} + \frac{\mu/\lambda \cdot 1/\lambda^2}{(1-\mu/\lambda)(1-1/\lambda^2)} + \frac{\lambda/\mu \cdot 1/\mu^2}{(1-\lambda/\mu)(1-1/\mu^2)} = \underline{1}$$

$$\Omega^{0,q}(X; K_X) = \Omega^{n,q}(X; \mathbb{C})$$

$$\lambda\mu = \operatorname{tr} f^* \Big|\, \underbrace{\operatorname{Res}_{\mathbb{C}^3|X} \frac{dx\wedge dy\wedge dz}{z^2-xy}}$$

Poincaré Residue

$$\operatorname{tr} f^*(z\cdot w) = \mu/\lambda \cdot 1/\lambda^2$$

$$\underbrace{\left[\frac{dx\wedge dy}{2z}\right]}_{\in L^2} \longrightarrow \frac{\lambda^2\mu^2}{\lambda\mu} = (\lambda\mu)$$

$\exists!\ L \longrightarrow X$ s.t. $L^{\otimes 2} = K_X$ "$L = \sqrt{K_X}$" $\cong$ Spin structure!

$$3]\quad \frac{1+\lambda\mu\sqrt{(\lambda\mu)}}{(1-\lambda^2)(1-\mu^2)} + \frac{\sqrt{\mu/\lambda \cdot 1/\lambda^2}}{(1-\mu/\lambda)(1-1/\lambda^2)} + \frac{\sqrt{\lambda/\mu \cdot 1/\mu^2}}{(1-\lambda/\mu)(1-1/\mu^2)} = 0 = \operatorname{Ind}(\not{\partial}_{spin})$$

Rigidity + Vanishing

4 $\quad p=1 \}$ $\quad \Omega^1(\mathcal{U}_a)$ <u>not a locally free module over $\mathcal{O}_X$</u>

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$$R = \mathbb{C}[[x,y]]$$

$$R\left[dx, dy, \underbrace{\frac{x\,dy}{z}, \frac{y\,dx}{z}}\right]$$

meromorphic but L^2 bounded!!

$$\frac{\lambda^2+\mu^2+\lambda\mu+\lambda\mu}{(1-\lambda^2)(1-\mu^2)} + \frac{\mu/\lambda + 1/\lambda^2}{(1-\mu/\lambda)(1-1/\lambda^2)} + \frac{\lambda/\mu + 1/\mu^2}{(1-\lambda/\mu)(1-1/\mu^2)} = -1$$

$$\chi_{y,b}(\theta) := \sum_{p=0}^{n} y^p \sum_{q=0}^{n} b^q \, \mathrm{Tr}\, T_\theta \big|_{H^{p,q}}$$

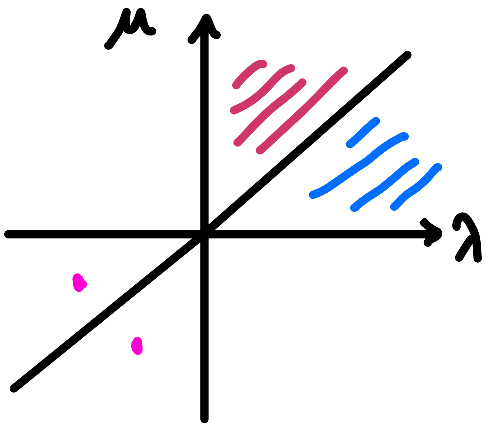
Poincaré Hodge polynomial (equivariant)

$$\chi_y(\theta) := \chi_{y,-1}(\theta)$$

Hirzebruch χ_y invariant (equivariant)

Morse polynomials:

$$\frac{1+\lambda\mu}{(1-\lambda^2)(1-\mu^2)} + \frac{1}{\left(1-\mu/\lambda\right)\left(1-1/\lambda^2\right)} + \frac{1}{\left(1-\lambda/\mu\right)\left(1-1/\mu^2\right)} = \frac{1}{\quad}$$



$$\lambda = e^{i\theta_1}, \quad \mu = e^{i\theta_2}$$

$$\frac{1+\lambda\mu}{(1-\lambda^2)(1-\mu^2)} + \frac{b^2\left(\lambda/\mu\right)\quad\left(\lambda^2\right)}{\left(1-\lambda/\mu\right)\left(1-\lambda^2\right)} + \frac{b\,\mu^2}{\left(1-\lambda/\mu\right)\left(1-\mu^2\right)} = 1 + (1+b)\,\Sigma$$

$$\frac{1+\lambda\mu}{(1-\lambda^2)(1-\mu^2)} + \frac{b\qquad\lambda^2}{\left(1-\mu/\lambda\right)\left(1-\lambda^2\right)} + \frac{b^2\left(\mu/\lambda\right)\ \left(\mu^2\right)}{\left(1-\mu/\lambda\right)\left(1-\mu^2\right)} = 1 + (1+b)\,\Sigma$$

Thm (J.)

Vanishing theorem for Harmonic spinors

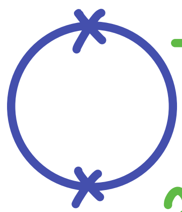
Given X normal toric variety with canonical singularities, wedge metric, admitting a spin structure, then $\nexists$ global harmonic spinors $\left(\mathcal{D}_{min}(\bar{\partial}^*_{\mathcal{K}^{1/2}})\right)$.

(Pf)

$\exists$ Key — $(\mathcal{K}^{1/2})^* \otimes \mathcal{K} = \mathcal{K}^{-1/2} \otimes \mathcal{K} = \mathcal{K}^{1/2}$

S.D. under S.D.

Eg —



$\dfrac{\lambda^{1/2}}{1-\lambda}$

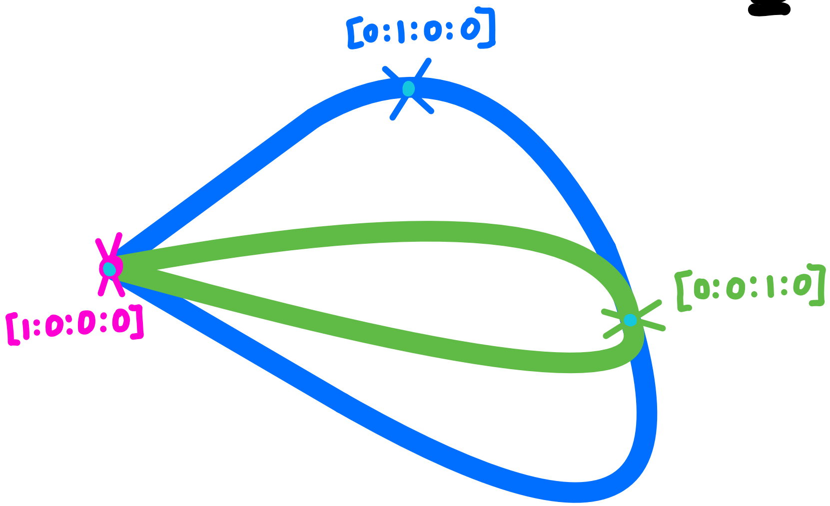
$\dfrac{\lambda^{-1/2}}{1-\lambda^{-1/2}}$

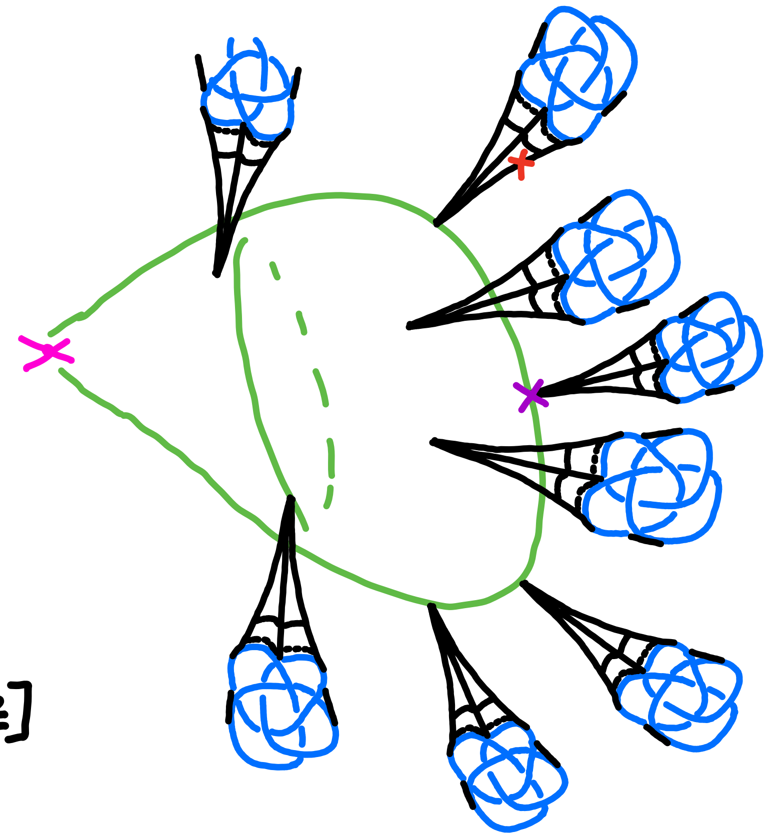
$$\frac{\lambda^{1/2}}{1-\lambda} + b\,\frac{\lambda^{1/2}}{1-\lambda}$$

$$\frac{b\,\lambda^{-1/2}}{1-\lambda^{-1}} + \frac{\lambda^{-1/2}}{1-\lambda^{-1}}$$

$\mathbb{CP}^3\,[W:X:Y:Z] \hookleftarrow Z^4 - X^3 Y = 0$

$\supseteq X = Z = 0 \supseteq X = Y = Z = 0$





$(\lambda, \mu)\cdot[W:X:Y:Z] = [W : \lambda^4 X : \mu^4 Y : \lambda^3 \mu Z]$

$\lambda = e^{i\theta_1} \quad \mu = e^{i\theta_2}$

$$Z^4 - X^3 Y = 0 \qquad (\lambda,\mu)\,[W:X:Y:Z] \longrightarrow [W:\lambda^4 X:\mu^4 Y:\lambda^3\mu Z]$$

$$\underbrace{\frac{1+(\lambda^3\mu)+(\lambda^3\mu)^2+(\lambda^3\mu)^3}{(1-\lambda^4)(1-\mu^4)}}_{\mathbb{C}[[x,y]][1,z,z^2,z^3]} + \frac{1}{(1-1/\lambda^4)(1-\mu/\lambda)} + \frac{1+(\lambda^3/\mu^3)+(\lambda^3/\mu^3)^2+(\lambda^3/\mu^3)^3}{(1-1/\mu^4)(1-\lambda^4/\mu^4)}$$

$$= 1 + \frac{\lambda^5}{\mu}$$

$$\underbrace{\frac{1+(\lambda^3\mu)+(\lambda^2\mu^2)+(\lambda\mu^3)}{(1-\lambda^4)(1-\mu^4)}}_{\text{trace over}} + \frac{1}{(1-1/\lambda^4)(1-\mu/\lambda)} + \frac{1}{(1-\lambda/\mu)(1-1/\mu^4)} = 1$$

trace over

$R_1 = R[1, z, x\underbrace{y/z}, x^2\underbrace{y/z^2}]$ holomorphic not regular

$xy/z \sim xy/(x^{3/4}y^{1/4})$

$R = \mathbb{C}[x,y]$

$x = \frac{X}{W}, \; y = \frac{Y}{W}, \; z = \frac{Z}{W}$

integral completion

VECTOR FIELDS AND CHARACTERISTIC NUMBERS

Raoul Bott

Theorem 1 is really a byproduct of M. Atiyah's and my work on the generalized fixed point theorem [1], and its history is as follows. A formula of this type was first conjectured to me by V. Guillemin, who derived some special cases of it from our fixed-point formula and the Riemann-Roch formula of Hirzebruch. Next, M. Atiyah pointed out that there really were sufficiently many such special cases to prove the theorem in general as a consequence of our fixed-point formula.

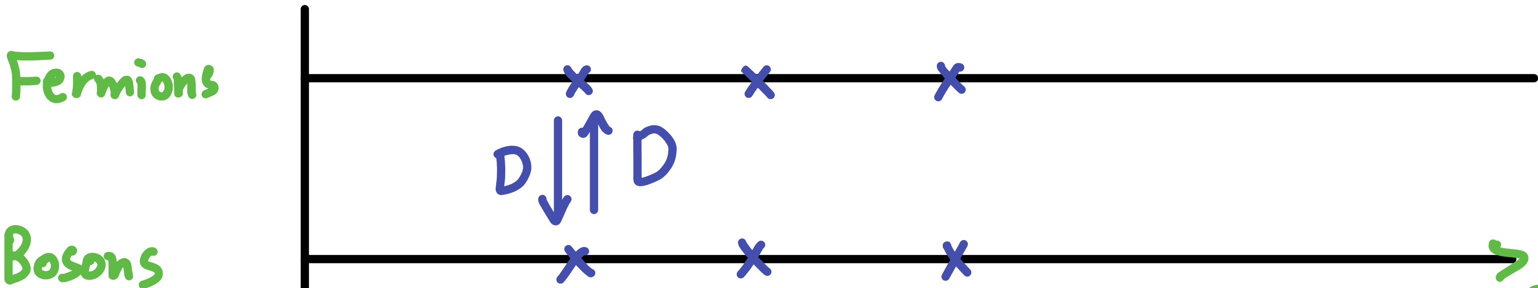
Applying this procedure to sufficiently many of the sheaves

$$\Omega_1^{a_1} \otimes \Omega_2^{a_2} \otimes \cdots \otimes \Omega_m^{a_m},$$

(where Ω_r denotes the sheaf of germs of holomorphic r-forms on X) and using some standard K-theory, one can derive sufficiently many verifications of Theorem 1 to prove it in general.

Any elliptic first order operator!

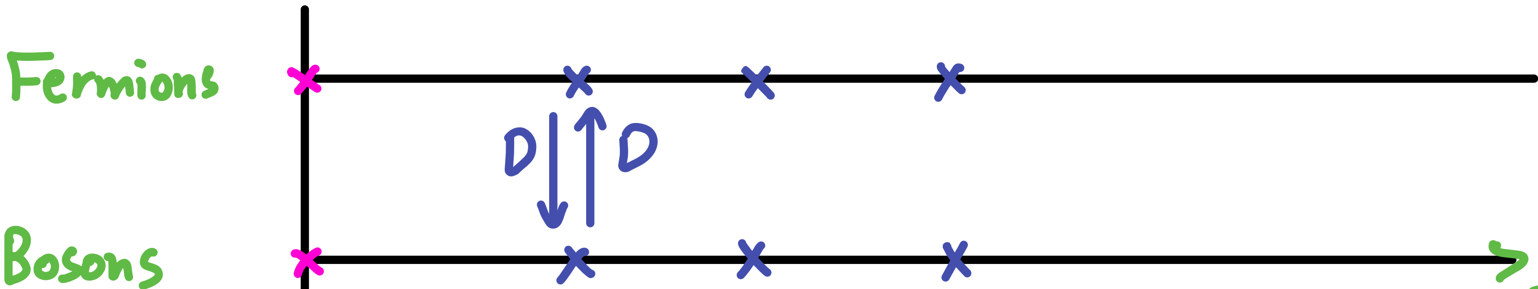
SUSY breaking!



Energy $= \lambda$

SUSY

$$D\varphi_\lambda = \sqrt{\lambda}\,\psi_\lambda$$



Energy

no SUSY

$$D\varphi_0 = 0$$

Fermion Quantum Numbers in Kaluza-Klein Theory

Edward Witten

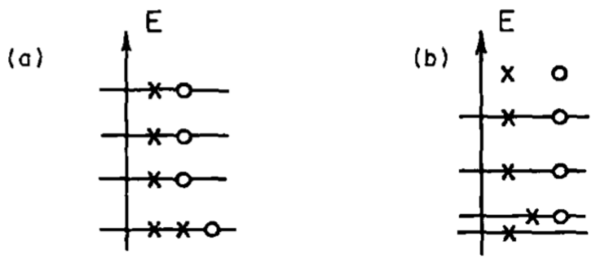
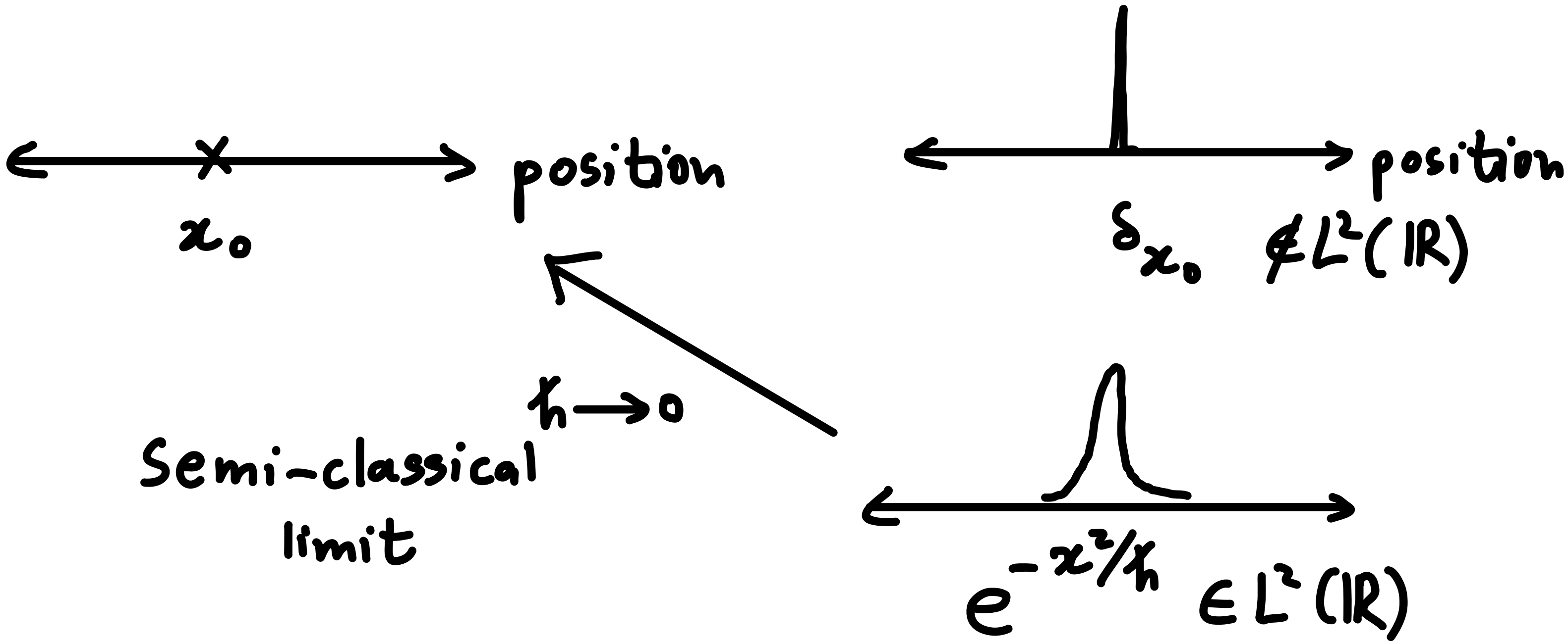


Figure 1
Zero modes of the Dirac operator of positive or negative chirality are indicated by $\times$ or $\bigcirc$, respectively. The number of $\times$s minus $\bigcirc$s at zero energy is invariant under perturbations.

and cancel out of the trace). The trace of a matrix is the sum of ... matrix elements. In the coordinate basis, the diagonal matrix elements of $e^{i\theta\mathcal{L}_K}$ vanish except near the zeros of K. The fixed point formula is similar to the method of Landau and Lifshitz[27] for computing the character of a molecular symmetry group furnished by the molecular vibrations in terms of fixed points of the symmetry group action.

To gain some practice with (35), let us use these methods to retrieve the results of section III. We consider a spin $\frac{1}{2}$ particle moving on the two-sphere. We take K to be (figure 5) the generator of a rotation about the z



$$z \in \frac{\ker \bar{\partial}}{\operatorname{im} \bar{\partial}} \qquad z\, e^{-|z|^2/\hbar} \in \frac{\ker \bar{\partial}_{1/\hbar}}{\operatorname{im} \bar{\partial}_{1/\hbar}}$$

SUSY breaking!

$$\therefore \quad \bar\partial \longrightarrow e^{-ith/\hbar}\,\bar\partial\, e^{ith/\hbar}$$

Unitary evolution of states/operators

$\therefore$ Wick rotate $t = -iT$

$$\bar\partial \longrightarrow e^{-Th/\hbar}\,\bar\partial\, e^{+Th/\hbar}$$

send $\hbar \longrightarrow 0$??

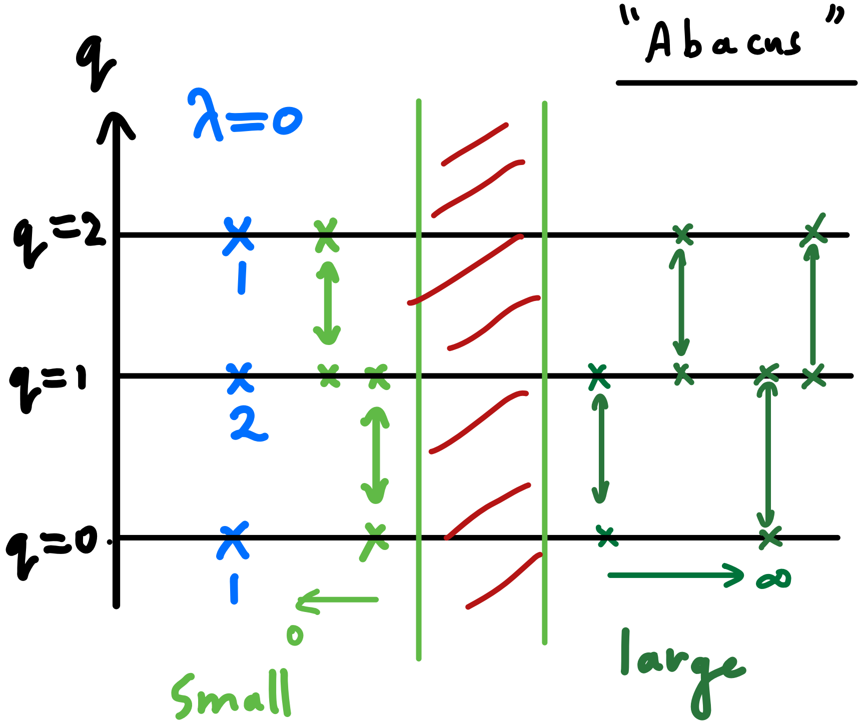
~~Unitary~~ ~~conjugation~~

$\because$ $\bar\partial \longrightarrow e^{-ith/\hbar}\,\bar\partial\, e^{ith/\hbar}$

$\because$ $\bar\partial \longrightarrow e^{-Th/\hbar}\,\bar\partial\, e^{+Th/\hbar} = \bar\partial_{1/\hbar}$ (say $T=1$)

$\because$ isomorphic as Hilbert complexes

$$\begin{tikzcd}
\arrow[r] & L^2\Omega^{0,q}(X;E) \arrow[r, "\bar\partial"] \arrow[d, "e^{-h/\hbar}"] & L^2\Omega^{0,q+1}(X;E) \arrow[r] \arrow[d, "e^{-h/\hbar}"] & {} \\
\arrow[r] & L^2\Omega^{0,q}(X;E) \arrow[r, "\bar\partial_{1/\hbar}"'] & L^2\Omega^{0,q+1}(X;E) \arrow[r] & {}
\end{tikzcd}$$

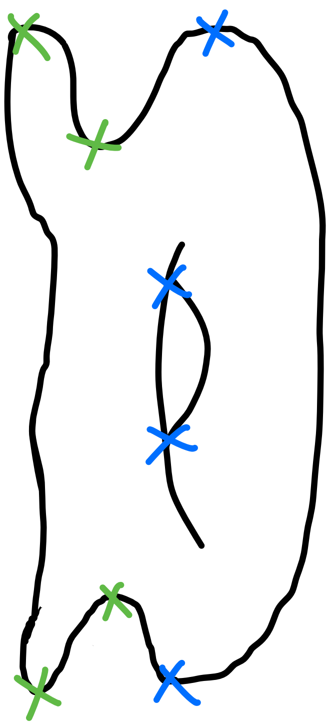


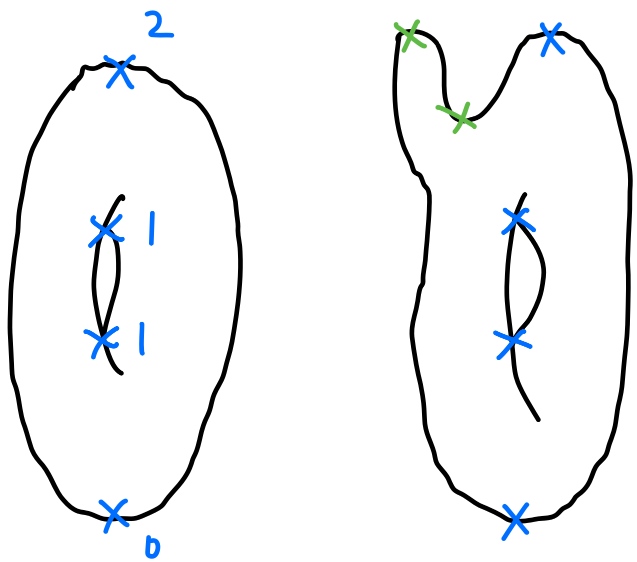
Morse polynomial(b)

$$= \underbrace{b^2 + 2b^1 + b^0}_{} + (b^2 + b^1) + (b^1 + b^0)$$

$$= \underbrace{\text{Poincaré polynomial}(b)}_{\sum_{q=0}^{n} b^q \dim \mathcal{H}(X)} + \text{Error polynomial}(b)$$

$$+ (1+b)\sum_{q=0}^{n} b^q \underbrace{Q_q}_{\text{Non-negative integer}}$$



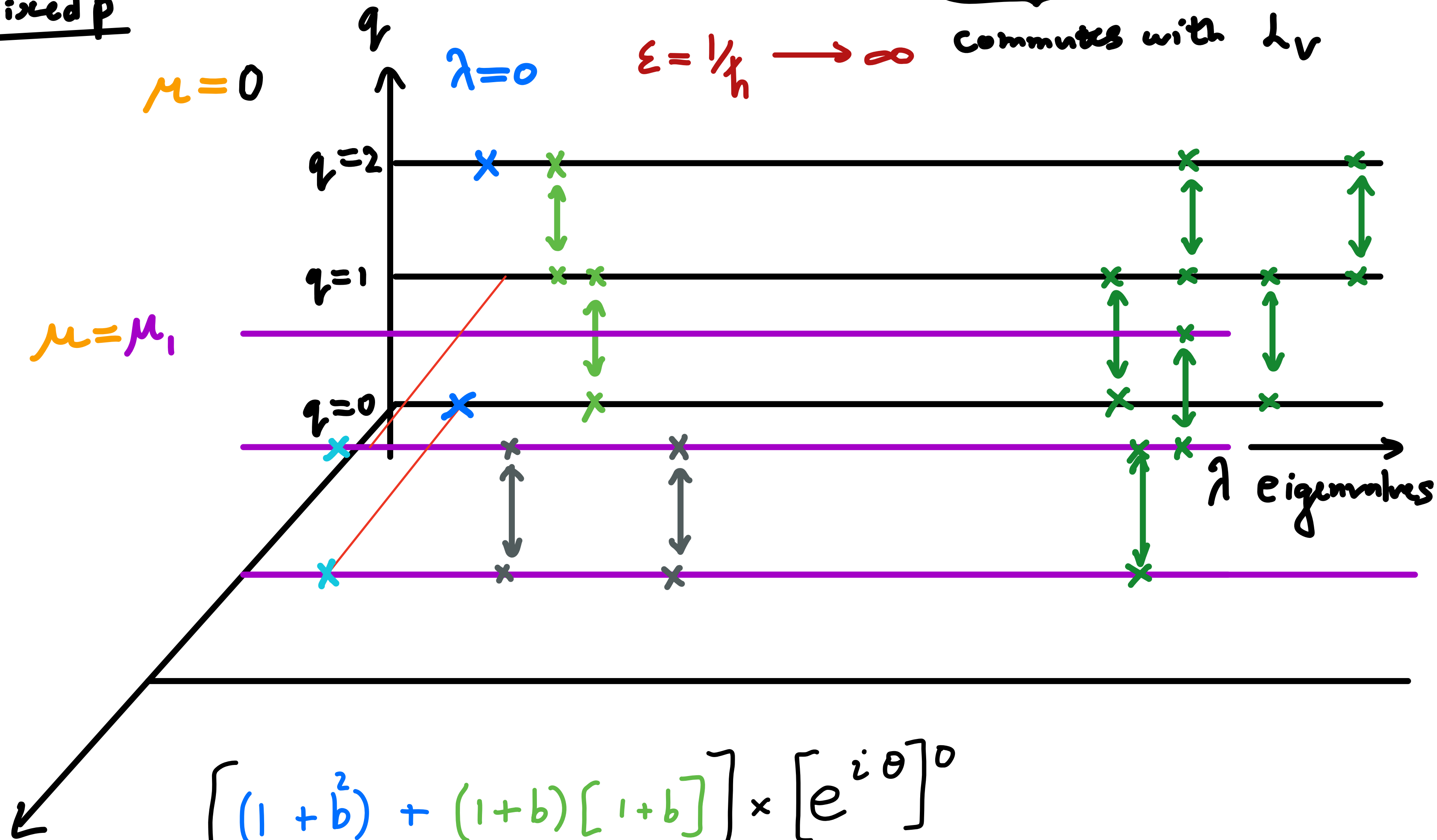


$\Omega^{p,q}(X;E)$

fixed p

$$\Delta_\varepsilon = \bar\partial_\varepsilon \bar\partial_\varepsilon^* + \bar\partial_\varepsilon^* \bar\partial_\varepsilon = \underbrace{d_\varepsilon d_\varepsilon^* + d_\varepsilon^* d_\varepsilon}_{\text{commutes with } L_V} + \varepsilon\sqrt{-1}\, L_V$$

$\varepsilon = 1/\hbar \longrightarrow \infty$



$$\left[(1+b^2) + (1+b)[1+b]\right] \times \left[e^{i\theta}\right]^0$$

$$+\left[(1+b) + (1+b)\times z\right] \times \left[e^{i\theta}\right]^{\mu_1}$$

$+$

Schrödinger operators & Morse theory

"Bochner type formulas" $\quad \Delta_{1/\hbar} = \bar\partial_{1/\hbar} \bar\partial^*_{1/\hbar} + \bar\partial^*_{1/\hbar} \bar\partial_{1/\hbar}$

$$\Delta_\varepsilon = \Delta + \varepsilon \underbrace{K} + \varepsilon^2 |dh|^2$$

$$\hbar^2 \Delta_{1/\hbar} = \Delta + \hbar \underbrace{K} + |dh|^2$$

0^th^-order bdd operator

GRAVITATIONAL ANOMALIES

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Edward WITTEN[2]
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544, USA

in ten dimensions. But now we meet a real surprise, which is by far the most striking result of this paper. The expressions for $\hat{I}_{1/2}$, $\hat{I}_{3/2}$, and $\hat{I}_A$ in (119) are linearly dependent. In addition, the minimal solution is remarkably simple: $-\hat{I}_{1/2}+\hat{I}_{3/2}+\hat{I}_A = 0$. Thus, a ten-dimensional theory with one (complex) negative chirality spin-$\frac{1}{2}$ field, one (complex) positive chirality spin-$\frac{3}{2}$ field, and one (real) self-dual antisymmetric tensor is free of anomalies. What is more, modulo fields that do not contribute anomalies, this is precisely the field content of the chiral $n=2$ supergravity theory in ten dimensions [11], which is the naive low-energy limit of one of the ten-

---

Spin-3/2

Gravitino

"The"

Rarita-Schwinger op.

$$(\bar\partial + \bar\partial^*)_E$$

$$E = (TX \otimes K^{1/2}) \oplus k\,(K^{1/2})$$

SUGY $k=-1$

(M-theory $k=-3$)

Classification of Gravitational Instanton Symmetries

G. W. Gibbons and S. W. Hawking

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 9EW, England

Acknowledgements. We should thank J. F. Adams, M. F. Atiyah, N. Hitchin, D. N. Page and C. N. Pope for help and discussions.

$$\tau = \sum_{\text{nuts}} -\cot\text{an}\, p\theta \cot\text{an}\, q\theta + \sum_{\text{bolts}} Y \operatorname{cosec}^2\theta \,, \tag{4.7}$$

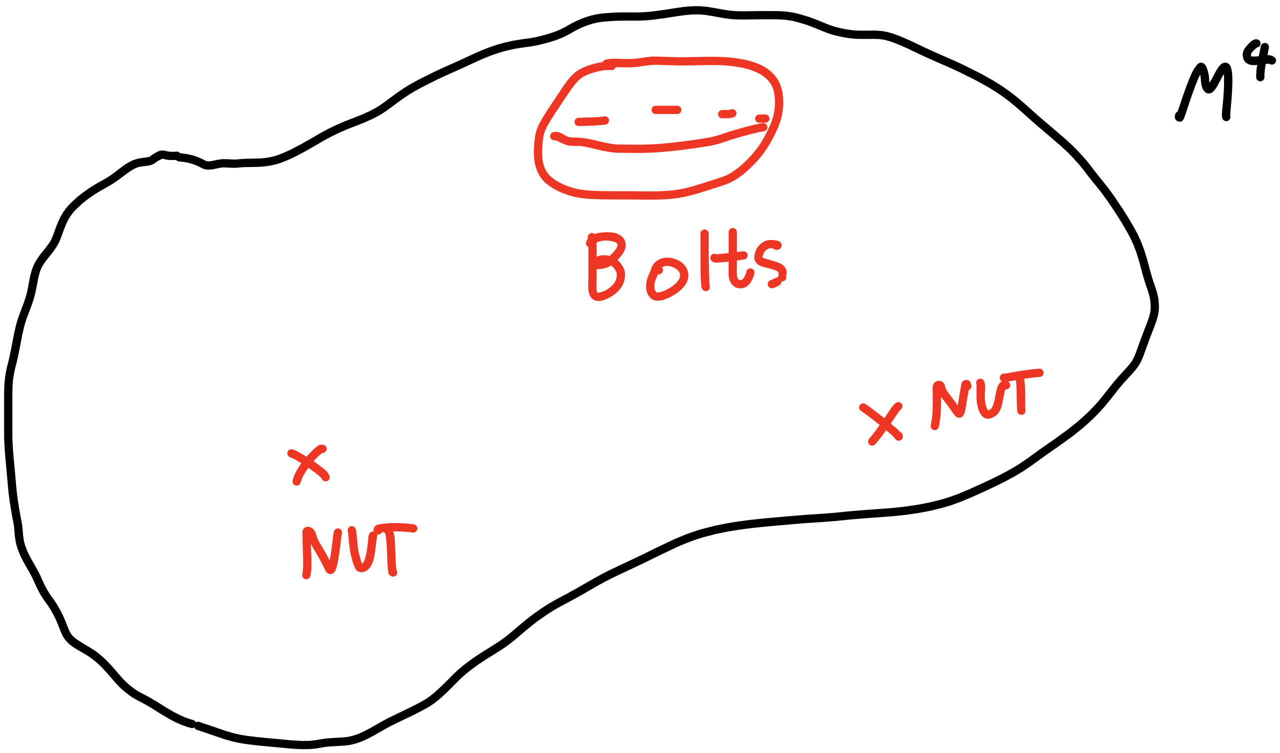
Y is the self intersection number of a bolt and 2θ is the group parameter.

Equation (4.7) holds for all values of θ. If one expands in powers of θ the first two terms give

$$\boxed{\sum_{\text{nuts}} -(pq)^{-1} + \sum_{\text{bolts}} Y = 0 \,,} \tag{4.8}$$

$$\sum_{\text{nuts}} \tfrac{1}{3}(pq^{-1} + qp^{-1}) + \tfrac{1}{3} \sum_{\text{bolts}} Y = \tau \,. \tag{4.9}$$

Applying (4.8) to the Killing vector $\dfrac{\partial}{\partial\psi}$ in CP^2 which has an antinut with



Fixed points of Killing vector field K

Classification of Gravitational Instanton Symmetries

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$$\boxed{\sum_{\text{nuts}} -(pq)^{-1} + \sum_{\text{bolts}} Y = 0 \,,} \; - \boxed{2} \qquad \text{Rigidity!!} \tag{4.8}$$

$$\sum_{\text{nuts}} \tfrac{1}{3}(pq^{-1} + qp^{-1}) + \tfrac{1}{3} \sum_{\text{bolts}} Y = \tau \,. \;\vdash \boxed{3} \longrightarrow \text{Classify} \tag{4.9}$$

Applying (4.8) to the Killing vector $\dfrac{\partial}{\partial \psi}$ in CP^2 which has an antinut with

$$\chi_{+1}(2\theta) = \underbrace{c_{-2}}_{\text{NUT charge}} (2\theta)^{-2} + c_0 + \sum_{k>0} c_k (2\theta)^k$$

5. Duality

Gibbons - Hawking Ansatz

The action of the group G with Killing vector $K = \frac{\partial}{\partial \tau}$ defines a fibering $\pi: M - C \to B$ where C is the fixed point set of μ_τ. In other words B is the 3-dimensional space of non-trivial orbits of G. The manifold B inherits a metric

$$h_{ab} = g_{ab} - V^{-1} K_a K_b \, , \tag{5.1}$$

where $V = K^a K_a$. The metric g_{ab} on M can then be written locally in the form

$$ds^2 = V(d\tau + \omega_i dx^i)^2 + V^{-1} \gamma_{ij} dx^i dx^j \, , \tag{5.2}$$

The twist field $H_{ij} = \partial_i \omega_j - \partial_j \omega_i$ is gauge invariant. It can be expressed as

$$H_{ab} = 2K_{e;f} h^e_b h^f_a V^{-1} \, . \tag{5.5}$$

For the rest of this section we shall work in the 3-dimensional space B. Indices i, j, k etc. will be raised or lowered by γ_{ij} and covariant differentiation with respect to γ_{ij} will be denoted by $\|$. Using the 3-dimensional alternating tensor one can define a twist vector H_i by

$$H_i = \tfrac{1}{2} \varepsilon_i^{jk} H_{jk} \, . \tag{5.6}$$

This obeys the conservation equation

$$H^i_{\|i} = 0 \, . \tag{5.7}$$

One can therefore define the nut charge within a 2-surface L by

$$N = (8\pi)^{-1} \int_L \underline{H_i d\sigma^i} \, . \quad = \frac{1}{8\pi} \int_L c_1 \tag{5.8}$$

One can therefore define the nut charge within a 2-surface L by

$$N=(8\pi)^{-1}\int_L H_i d\sigma^i .$$

In the case of a nut of type (p,q)

$$N=(8\pi pq)^{-1}\beta .$$

For a bolt with self intersection number Y

$$N=(8\pi)^{-1}Y\beta .$$

A_n is the area of the n'th bolt. Thus

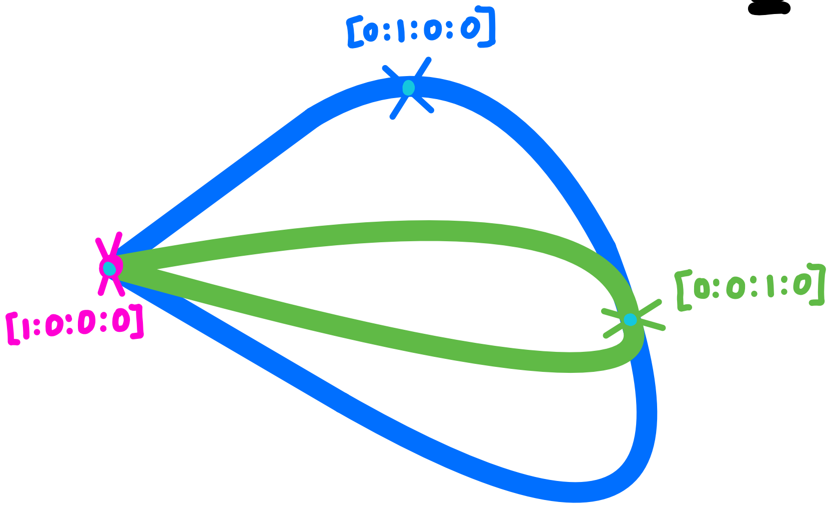
$$\begin{aligned}\hat{I} &= \sum_{\text{bolts}} -\tfrac{1}{4}A_n + \sum_{\text{bolts}} \tfrac{1}{2}\psi_n N_n\beta + \sum_{\text{nuts}} -\tfrac{1}{2}\psi_n N_n\beta \\ &= \sum_{\text{bolts}} -\tfrac{1}{4}A_n + \sum_{\text{bolts}} \frac{\psi Y\beta^2}{16\pi} - \sum_{\text{nuts}} \frac{\psi\beta^2}{16\pi pq} .\end{aligned}$$

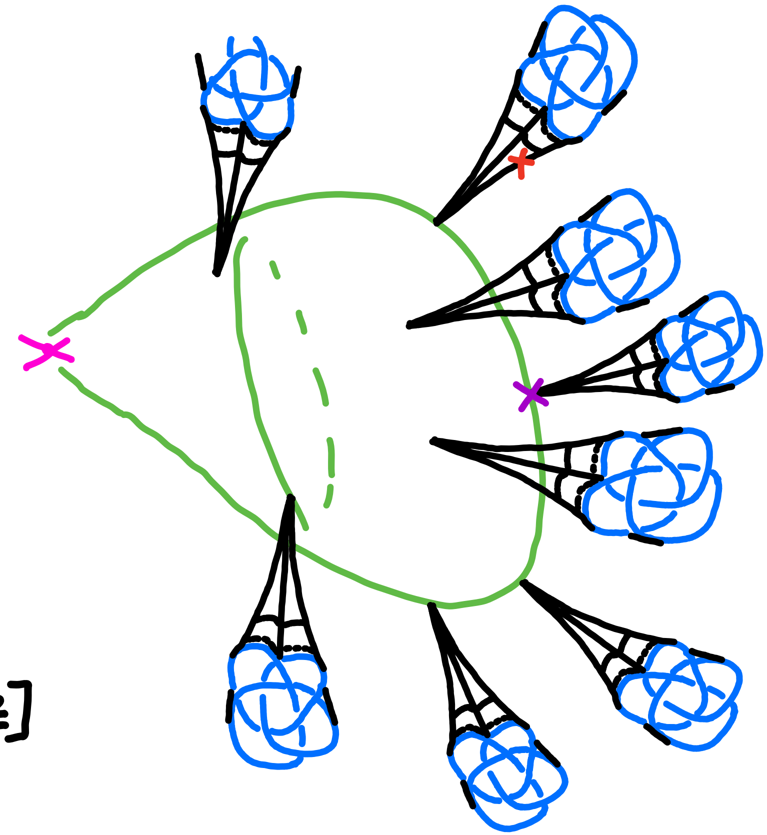
This generalizes the formula

$$\hat{I} = -\tfrac{1}{4}A$$

$$\mathbb{CP}^3[W:X:Y:Z] \hookleftarrow Z^4 - X^3 Y = 0$$

$$\supseteq X = Z = 0 \supseteq X = Y = Z = 0$$





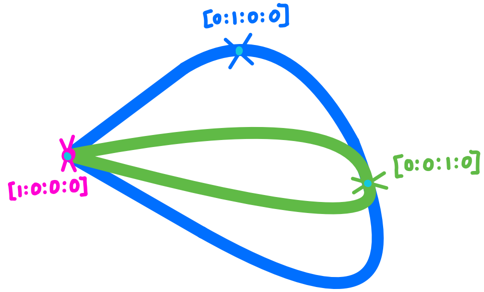
$$(\lambda, \mu) \cdot [W : X : Y : Z] = [W : \lambda^4 X : \mu^4 Y : \lambda^3 \mu Z]$$

$$\lambda = e^{i\theta_1} \quad \mu = e^{i\theta_2}$$

i] Generic $\theta_1, \theta_2 \Longrightarrow$ 3 NUTS

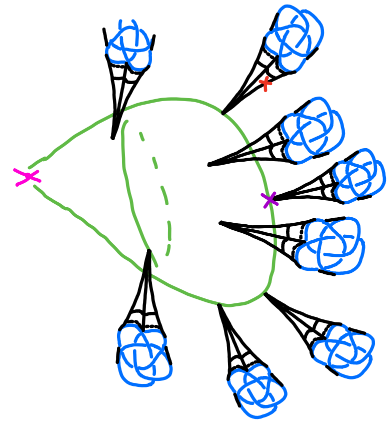
ii] $\theta_2 = 0 \longrightarrow$ 1 NUT, 1 BOLT

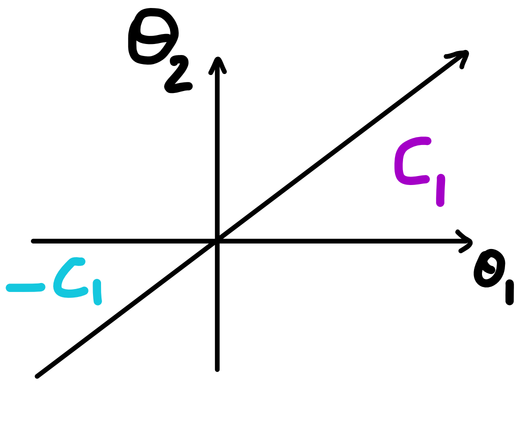
$$\mathbb{CP}^3[W:X:Y:Z] \hookleftarrow Z^4 - X^3Y = 0 \supseteq X = Z = 0 \supseteq X = Y = Z = 0$$



$$\mathrm{Ell}_{y,b} = \chi_{y,b}$$

$$= 1 + yb + y^2b^2$$





$$(\lambda,\mu)\cdot[W:X:Y:Z] = [W:\lambda^4X:\mu^4Y:\lambda^3\mu Z]$$

$$\lambda = e^{i\theta_1} \quad \mu = e^{i\theta_2}$$

set $\lambda = \mu^2 = e^{i\theta}$

i.e. – $\theta = \theta_1 = 2\theta_2$

$$\frac{1+\lambda^3\mu+\lambda^2\mu^2+\lambda^3\mu^3+y^1(\lambda+\mu)^2(\lambda^2+\mu^2)+y^2(\lambda^4\mu^4+\lambda\mu^3+\lambda^2\mu^2+\lambda^3\mu)}{(1-\lambda^4)(1-\mu^4)} + \frac{b^1\mu^4\left[1+y(\mu^{-4}+\lambda\mu^{-1})+y^2\mu^{-5}\lambda\right]}{(1-\mu^4)(1-\lambda\mu^{-1})}$$

$$+\frac{b^2\lambda^5\mu^{-1}\left[1+y(\lambda^{-4}+\mu\lambda^{-1})+y^2\lambda^{-5}\mu\right]}{(1-\lambda^4)(1-\mu^{-1}\lambda)} = \chi_{y,b}(2\theta)$$

Poincaré – Hodge