On the twisted Ruelle zeta function and the Ray-Singer metric

Polyxeni Spilioti

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Department of Mathematics
National and Kapodistrian University of Athens
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I. Introduction

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- Analogy: Riemann zeta function

$$\zeta(s) = \prod_{p = \text{prime}} (1 - p^{-s})^{-1}, \quad \mathrm{Re}(s) > 1$$

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$$|R(0;\rho)| = T^{RS}(X,\rho)^2,$$

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- harmonic analysis on locally symmetric spaces
- representation theory, Selberg trace formula

II. Fried's conjecture and non-unitary representations

- $X = \Gamma\backslash G/K$ is a d-dimensional locally symmetric compact hyperbolic manifold
- $G = \mathrm{SO}^0(d,1)$, $K = \mathrm{SO}(d)$, $d = 2n+1, n \in \mathbb{N}_>$
- $\widetilde{X} := G/K \cong \mathbb{H}^d$ using the Killing form
- Γ discrete, cocompact, torsion-free subgroup of G

- Fix notation
  - $\mathfrak{g}$ =Lie algebra of G
  - $\mathfrak{k}$ =Lie algebra of K
  - $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, the Cartan decomposition of $\mathfrak{g}$
  - $\mathfrak{a}$ = a maximal abelian subalgebra of $\mathfrak{p}$
  - A subgroup of G with Lie algebra $\mathfrak{a}$
  - $M := \mathrm{Centr}_K(A) = \mathrm{SO}(d-1)$

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- Γ cocompact $\dashrightarrow$ every $\gamma \in \Gamma$, with $\gamma \neq e$ is hyperbolic.
- then, $\gamma \sim_G m_\gamma a_\gamma$ (Wallach 1976).
- $\chi \colon \Gamma \to \mathrm{GL}(V_\chi)$ a finite dimensional representation of Γ, $\sigma \in \widehat{M}$.

Definition (Twisted Selberg zeta function)

$$Z(s;\sigma,\chi) := \prod_{\substack{[\gamma]\neq e \\ [\gamma]\text{ prime}}} \prod_{k=0}^{\infty} \det\Big(\mathrm{Id} - (\chi(\gamma)\otimes\sigma(m_\gamma)\otimes S^k(\mathrm{Ad}(m_\gamma a_\gamma)|_{\overline{\mathfrak{n}}}))e^{-(s+|\rho|)l(\gamma)} \Big),$$

for $\mathrm{Re}(s) > r$, r positive constant.

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for $\mathrm{Re}(s) > c$, c positive constant.

- The product runs over the prime conjugacy classes $[\gamma]$ of Γ, which correspond to the prime closed geodesics on X of length $l(\gamma)$.

Fried's conjecture:

("Lefshetz formulas for flows") Relate the Ruelle zeta function at zero with the analytic torsion.
$\rightsquigarrow$ By Cheeger-Mueller theorem one can pass to the Reidemeister torsion, a topological invariant.

- Fried 1986: Ruelle zeta function $R(s;\chi)$, associated with an orthogonal representation χ of the fundamental group. The leading term of the Laurent expansion of $R(s;\chi)$ at $s = 0$ is

$$C(\chi) T^{RS}(X;\chi)^2 s^h$$

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Related work:

- Bunke-Olbrich, Wotzke, Müller, Pfaff, Fedosova, Shen.
- Dang-Guillarmou-Riviere-Shen, Chaubet-Dang, Dyatlov-Delarue-Paternain, Dyatlov-Zworski, Abdurrahman-Venkatesh.

Analogies in number theory

$\rightsquigarrow$ Dedekind zeta function

$$\zeta_K(s) := \prod_{\mathfrak{p} \text{ prime} \subseteq \mathcal{O}_K} \left(1 - (N_{K/\mathbb{Q}}(\mathfrak{p}))^{-s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

$\rightsquigarrow$ Artin L-function associated with a Galois representation (twisted version)

$\rightsquigarrow$ Analytic class number formula $\zeta_K(s)$ has a simple pole at $s = 1$ with residue

$$\lim_{s \to 1} (s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} \operatorname{Reg}_K h_K}{w_K \sqrt{|D_K|}}$$

(a reference: Morishita "Knots and primes").

Arbitrary representations of Γ on a f.d. complex vector space

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- Problem: there is no Hermitian metric h^χ, which is compatible with the flat connection on the flat vector bundle $\dashrightarrow$ we deal with non self-adjoint Laplace and Dirac operators
- Solution: Twisted Bochner-Laplace operator $\Delta^\sharp_{\tau,\chi}$ acting on smooth sections of twisted vector bundles $E_\tau \otimes E_\chi$, $\tau \in \widehat{K}$
  - it is an elliptic operator $\dashrightarrow$ nice spectral properties $\dashrightarrow$ its spectrum is discrete and contained in a translate of a positive cone in $\mathbb{C}$.
  - Consider the corresponding heat semi-group $e^{-t\Delta^\sharp_{\tau,\chi}}$. It is an integral operator with smooth kernel.
  - Consider the trace of the operator $e^{-t\Delta^\sharp_{\tau,\chi}}$ and derive a trace formula.

Trace formula

Theorem (S., 2018)

For every $\sigma \in \widehat{M}$ we have

$$\operatorname{Tr}(e^{-tA^{\sharp}_{\chi}(\sigma)}) = \dim(V_\chi)\operatorname{Vol}(X)\int_{\mathbb{R}} e^{-t\lambda^2} P_\sigma(i\lambda)d\lambda + \sum_{[\gamma]\neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} L(\gamma;\sigma,\chi)\frac{e^{-l(\gamma)^2/4t}}{(4\pi t)^{1/2}};$$

where

$$L(\gamma;\sigma,\chi) = \frac{\operatorname{tr}(\chi(\gamma)\otimes\sigma(m_\gamma))e^{-|\rho| l(\gamma)}}{\det(\operatorname{Id}-\operatorname{Ad}(m_\gamma a_\gamma)_{\overline{n}})}.$$

Meromorphic continuation for arbitrary representations of Γ

Theorem (S., 2018)

Let $\sigma \in \widehat{M}$ and $\chi\colon \Gamma \to \mathrm{GL}(V_\chi)$ be a finite dimensional representation of Γ. The twisted Selberg zeta function $Z(s;\sigma,\chi)$ associated with σ and χ admits a meromorphic continuation to the whole complex plane $\mathbb{C}$. Its singularities are described in terms of the discrete eigenvalues of the twisted operators $A^\sharp_\chi(\sigma)$ and $D^\sharp_\chi(\sigma)$ and their orders are described by the corresponding algebraic multiplicities.

Theorem (S., 2018)

For every $\sigma \in \widehat{M}$ and for every finite dimensional representation χ of Γ, the twisted Ruelle zeta function $R(s;\sigma,\chi)$ admits a meromorphic continuation to the whole complex plane $\mathbb{C}$.

Theorem (S., 2020)

Let χ *be a finite-dimensional complex representation of* Γ. *Let* $\Delta^{\sharp}_{\chi,k}$ *be the flat Hodge Laplacian, acting on the space of* k*-differential forms on* X *with values in the flat vector bundle* E_{χ}. *Then, the Ruelle zeta function has the representation*

$$R(s;\chi) = \prod_{k=0}^{d-1}\prod_{p=k}^{d-1} \det\nolimits_{\mathrm{gr}}\left(\Delta^{\sharp}_{\chi,k} + s(s+2(|\rho|-p))\right)^{(-1)^p} \cdot \exp\left((-1)^{\frac{d-1}{2}+1}\pi(d+1)\dim(V_{\chi})\frac{\mathrm{Vol}(X)}{\mathrm{Vol}(S^d)}s\right),$$

where $\mathrm{Vol}(S^d)$ *denotes the volume of the* d*-dimensional Euclidean unit sphere. Let* $d_{\chi,k} := \dim\mathrm{Ker}(\Delta^{\sharp}_{\chi,k})$. *Then, the singularity of the Ruelle zeta function at* $s=0$ *is of order*

$$\sum_{k=0}^{(d-1)/2}(d+1-2k)(-1)^k d_{\chi,k}.$$

Back to Fried's conjecture:

- Prove that the Ruelle zeta function is regular at 0 and is related to the refined analytic torsion as it is introduced by Braverman and Kappeler and the Cappell-Miller torsion defined by Cappell and Miller. This is rather a complex refinement of the Ray-Singer analytic torsion.

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- Problem: Hodge theory is not applicable:

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- Solution: suitable set of representations: deformations of acyclic and unitary representations of the fundamental group.

Let $V \subset \mathrm{Rep}(\pi_1(X), \mathbb{C}^n)$ be an open neighbourhood (in classical topology) of the set $\mathrm{Rep}_0^u(\pi_1(X), \mathbb{C}^n)$ of acyclic, unitary representations such that, for all $\chi \in V$, B_χ is bijective. B_χ is the odd signature operator, $B_{k,\chi}^2 = \Delta_{\chi,k}^\sharp$.

Theorem (S., 2020)

Let $\chi \in V$. Then, the Ruelle zeta function $R(s;\chi)$ is regular at $s = 0$ and is equal to the complex Cappell-Miller torsion,

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- See also results and extensions by Müller and Shen

Cappell-Miller torsion

- it is defined in terms of the bicomplex $(\Lambda^*(X,E), \partial, \partial^{*,\sharp})$: a combination of the

  $$\text{torsion}(\Lambda^k_{[0,\lambda]}(X,E), \partial, \partial^{*,\sharp}) \in \det(H^*(X,E)) \otimes \det(H^*(X,E))$$

  and the square of the Ray-Singer term

  $$\prod_{k=0}^{d} \det_\theta(\Delta^\sharp|_{\Lambda^k_{(\lambda,\infty)}(X,E)})^{k(-1)^{k+1}} \in \mathbb{C}$$

- flat Laplacian $\Delta^\sharp$ acting on $\Lambda^*(X,E)$ is given by

  $$\Delta^\sharp := \partial\partial^{*,\sharp} + \partial^{*,\sharp}\partial$$

Ray-Singer metric

- Open problem: General representation of Γ?
- Solution: Consider the Ray-Singer norm for the refined analytic torsion.
- As it is was first introduced by Quillen for Riemann surfaces. It is a norm on the determinant line of $H^\bullet(X, E)$.

Definition

The Ray-Singer metric on $\det(H^\bullet(X, E))$ is defined by

$$\| \cdot \|^{\mathrm{RS}}_{\det(H^\bullet(X,E))} : = \| \cdot \|_\lambda \cdot T^{\mathrm{RS}}_{(\lambda,\infty)}, \lambda \geq 0.$$

Theorem (S., 2024)

Let $X = \Gamma\backslash\mathbb{H}^d$ be a compact, oriented, odd-dimensional hyperbolic manifold. Let $\chi\colon \Gamma \to \mathrm{GL}(V_\chi)$ be a finite-dimensional, complex representation of Γ. Let h_k be the dimension of the generalized eigenspace of the twisted Hodge Lappace operator $\Delta^\sharp_{\chi,k}$ with eigenvalue zero. Let

$$h_0 = \sum_{k=0}^{d-1}(d+1-k)(-1)^k h_k$$

Then,

$$\left|\lim_{s\to 0} s^{-h_0} R(s;\chi)\right| = C(d,\chi)\left(\frac{\|\rho_{\mathrm{an}}(X;\chi)\|^{RS}_{\det(H^*(X,E_\chi))}}{\|\rho_{\mathrm{an}}(X;\chi)\|_0}\right)^2.$$

III. Further applications

Theorem (Harish-Chandra)

The set of the discrete series representations of G is not empty if and only if $\mathrm{rank}_{\mathbb{C}}(G) = \mathrm{rank}_{\mathbb{C}}(K)$.

Compact hyperbolic surface X

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- Twisted Ruelle zeta function

$$R(s;\chi) = \prod_{\substack{[\gamma]\neq e \\ [\gamma]\text{ prime}}} \det(\mathrm{Id} - \chi(\gamma) e^{-sl(\gamma)}), \quad \mathrm{Re}(s) > c_2$$

Since

$$R(s;\chi) = \frac{Z(s;\chi)}{Z(s+1;\chi)}$$

Then,

Corollary (Frahm-S., 2021)

The twisted Ruelle zeta function $R(s;\chi)$ admits a meromorphic continuation to $\mathbb{C}$.

Corollary (Frahm-S., 2021)

The twisted Ruelle zeta function $R(s;\chi)$ near $s=0$ is given by

$$R(s;\chi) = \pm(2\pi s)^{\dim(V_\chi)(2g-2)} + \textit{higher order terms.}$$

Here g is the genus of the surface.

Compact hyperbolic orbisurface X: connection to topological torsion but for the Seifert fiber space $X_1 = \Gamma \backslash \mathrm{PSL}_2(\mathbb{R}) = S(X)$ over X

$\rightsquigarrow$ exact sequence

$$1 \to \mathbb{Z} = \pi_1(\mathrm{PSO}(2)) \to \pi_1(X_1) \to \Gamma = \pi_1(X) \to 1$$

Theorem (Bénard-Frahm-S., 2021)

For any irreducible representation $\rho\colon \pi_1(X_1) \to \mathrm{GL}(V_\rho)$*, the Ruelle zeta function* $R(s;\rho)$ *converges on some right half plane in* $\mathbb{C}$ *and extends meromorphically to the whole complex plane. Moreover: If* $\rho(t) \neq \mathrm{Id}_{V_\rho}$*, then the representation* ρ *is acyclic. Let* $\mathfrak{e}_{geod}$ *be the Euler structure induced by the geodesic flow on* X_1*. Then*

$$R(0;\rho) = \pm\, \mathrm{tor}(X_1, V_\rho, \mathfrak{e}_{geod}),$$

where $\mathrm{tor}(X_1, V_\rho, \mathfrak{e}) \in \mathbb{C}^\times$ *denotes the Reidemeister–Turaev torsion of* X_1 *in the representation* V_ρ *and the Euler structure* $\mathfrak{e}_{geod}$*.*

Selberg trace formula with non-unitary twist for surfaces

$$\operatorname{tr}(e^{-t\Delta_\chi^\sharp}) = \frac{1}{4\pi^2}\dim(V_\chi)\operatorname{Vol}(X)\int_{\mathbb{R}} e^{-t(\lambda^2+\frac{1}{4})}\lambda\pi\tanh\lambda\pi\, d\lambda$$
$$+\sum_{[\gamma]\neq e}\operatorname{tr}\chi(\gamma)\frac{l(\gamma)}{n_\Gamma(\gamma)D(\gamma)}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-t(\lambda^2+\frac{1}{4})}e^{-il(\gamma)\lambda}d\lambda.$$

Selberg trace formula with non-unitary twist for orbisurfaces

$$
\begin{aligned}
&\operatorname{Tr}\left(e^{-tA^{\sharp}_{\tau_m,\rho}}\right) \\
&\quad = \frac{\operatorname{Vol}(X)\dim(V_\rho)}{4\pi}\left[\int_{\mathbb{R}} e^{-t\lambda^2}\frac{\lambda\sinh 2\pi\lambda}{\cosh 2\pi\lambda+\cos\pi m}\,d\lambda \right.\\
&\qquad\qquad\qquad \left. + \sum_{\substack{1\le\ell<|m| \\ \ell \text{ odd}}} (|m|-\ell)e^{\left(\frac{|m|-\ell}{2}\right)^2 t}\right] \\
&\quad + \frac{1}{2\sqrt{4\pi t}}\sum_{[\gamma]\text{ hyp.}} \frac{l(\gamma)\operatorname{tr}\rho(\gamma)}{n_\Gamma(\gamma)\sinh\frac{l(\gamma)}{2}} e^{-\frac{l(\gamma)^2}{4t}} \\
&\quad + \sum_{[\gamma]\text{ ell.}} \frac{\operatorname{tr}\rho(\gamma)}{4M(\gamma)\sin\theta(\gamma)}\left[\int_{\mathbb{R}} e^{-t\lambda^2}\frac{\cosh 2(\pi-\theta(\gamma))\lambda + e^{i\pi m}\cosh 2\theta(\gamma)\lambda}{\cosh 2\pi\lambda+\cos\pi m}\,d \right.\\
&\qquad\qquad\qquad \left. + 2i\operatorname{sign}(m)\sum_{\substack{1\le\ell<|m| \\ \ell \text{ odd}}} e^{i\operatorname{sign}(m)(|m|-\ell)\theta(\gamma)}e^{\left(\frac{|m|-\ell}{2}\right)^2 t}\right]
\end{aligned}
$$

Further applications: Bottom of the spectrum, Teichmüller representations

- Define the critical exponent δ of the representation ϱ by

$$\delta(\varrho) := \inf\left\{ s > 0 \ :\ \sum_{\gamma\in\mathcal{P},k\geq 1} |\mathrm{tr}(\varrho(\gamma^k))| e^{-sk\ell(\gamma)} < \infty \right\}.$$

- Define the parabolic regions $\mathcal{C}_\sigma \subset \mathbb{C}$ for all $\sigma \in (1/2, +\infty)$ by

$$\mathcal{C}_\sigma := \left\{ z = x + iy \in \mathbb{C} \ :\ x \geq \sigma(1-\sigma) + \frac{y^2}{(1-2\sigma)^2} \right\}.$$

- $P_\sigma := \partial\mathcal{C}_\sigma$, the parabola given by the equation $\{x + iy \ :\ x = \sigma(1-\sigma) + \frac{y^2}{(1-2\sigma)^2}\}$.

Theorem (Naud-S., 2022)

Let Δ_ϱ be the twisted Laplacian defined as above, then one always have

$$\mathrm{Sp}(\Delta_\varrho) \subset \mathcal{C}_{\delta(\varrho)}.$$

Projects

1. Locally symmetric spaces of real rank 1.
2. Consider not necessarily the deformations of the acyclic and unitary representations as before.

Open problems: the cofinite case ? $\rightsquigarrow$ arithmetic manifolds.

Thank you