Harmonic maps between pseudo-Riemannian surfaces

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Outline

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Harmonic maps

M, N Riemannian manifolds, $u\colon M \to N$

$$e(u) = Tr_g(u^*(h)) = g^{ij}(x)h_{\alpha\beta}(u(x))\partial_i u^\alpha \partial_j u^\beta .$$

$$E(u) = \int_M e(u) dv_M ,$$

Euler-Lagrange equations of the energy functional are ta nonlinear system of elliptic PDEs of second order.
Solutions: Harmonic Maps

Riemann surfaces

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Mapping between surfaces: Isothermal coordinates...

$$g = e^{f(z,\bar{z})}|dz|^2, h = e^{F(u,\bar{u})}|du|^2$$

$$E(u) = \int e^{F(u,\bar{u})}(|\partial_z u_z|^2 + |\partial_{\bar{z}} u|^2)\, dxdy,$$

Harmonic map equations:

$$\partial_{z\bar{z}} u + \partial_u F(u,\bar{u}) \partial_z u \partial_{\bar{z}} u = 0.$$

Integrability

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$u = R + iS$ harmonic iff

$$\partial_{\bar{z}} \left(e^{F(u,\bar{u})} \partial_z u \partial_z \bar{u} \right) = 0$$

Consider locally a conformal change of coordinates s.t.

$$e^{F(u,\bar{u})} \partial_z u \partial_z \bar{u} = 1$$

Change of coordinates

If

$$e^{F(u,\bar{u})}\partial_z u \partial_z \bar{u} = e^{-\lambda(z)}$$

then

$$\zeta = \xi + i\eta = \int e^{-\lambda(z)/2}\, dz$$

Parametrization

Set

$$e^{v} = e^{\frac{F}{2}}\partial_z u \text{ and } v = \omega + i\theta.$$

Then

$$e^{v} = e^{\frac{F}{2}}\partial_z u$$

$$e^{-\bar{v}} = e^{\frac{F}{2}}\partial_{\bar{z}} u$$

Observation

solutions of Beltrami equation

$$\frac{\partial_{\bar{z}} u}{\partial_z u} = e^{-2\omega}$$

Compatability conditions

Since $\partial^2_{z\bar{z}} v = \partial^2_{\bar{z}z} v$ we get:

$$\partial_x \omega - \partial_y \theta = e^{-\frac{F(R,S)}{2}} \sinh \omega (\partial_R F \cos \theta + \partial_S F \sin \theta)$$

$$\partial_x \theta + \partial_y \omega = e^{-\frac{F(R,S)}{2}} \cosh \omega (\partial_S F \cos \theta - \partial_R F \sin \theta)$$

sinh-Gordon

Since $\partial^2_{xy}\theta = \partial^2_{yx}\theta$ we get:

$$\Delta\omega = 2e^{-F}\Delta F \sinh\omega\cosh\omega = -2K_N \sinh 2\omega$$

Bäcklund transformation

$\Delta\omega = -2K_N \sinh(2\omega)$, Bäcklund transformation

$i\Delta\theta = 2e^{-F}\left(\left(F_u^2 - F_{uu}\right)e^{2i\theta} - \left(F_{\bar{z}}^2 - F_{\bar{z}\bar{z}}\right)e^{-2i\theta}\right)$

Bäcklund transformation

Let

$$e^F = \frac{1}{S^2}.$$

$$\partial_x \omega - \partial_y \theta = -2 \sinh \omega \sin \theta$$

$$\partial_x \theta + \partial_y \omega = -2 \cosh \omega \cos \theta$$

Bäcklund transformation:

$$\Delta \omega = -2 \sinh (2\omega) \leftrightarrow \Delta \theta = -2 \sin (2\theta)$$

Constant Curvature: Uniform Approach

We have to solve sinh-Gordon

$$\Delta\omega = -2K_N \sinh 2\omega$$

and Beltrami

$$\frac{\partial_{\bar{z}} u}{\partial_z u} = e^{-2\omega}$$

Assume

$$\omega = \omega(ax + by)$$

one soliton solution...

Elliptic functions

$$x = \int_0^{\phi} \frac{1}{\sqrt{1 - m\sin^2 t}} dt$$

$$sn(x, m) = \sin\phi$$

$$cn(x, m) = \cos\phi$$

$$dn(x, m) = \sqrt{1 - m\sin^2 x}$$

$$\Pi(n, \phi, m) = \int_0^{\phi} \frac{1}{(1 - n\sin^2 t)\sqrt{1 - m\sin^2 t}} dt$$

1-soliton solutions

Solve sinh-Gordon equation when $K_N = const$ and $\omega = \omega(ax + by)$:

$$\Delta\omega = -2K_N \sinh 2\omega$$

via elliptic integrals

1-soliton solutions

$$R - R_0 = X - X_0 + i\frac{\tan\tau}{c}\left(\Pi(\frac{1}{\cos^2\tau}, iw, -\frac{4K_N}{c\rho^2}) - \Pi_0\right)$$

$$S - S_0 = \frac{1}{\sqrt{c}\sqrt{1+\frac{4K_N}{c\rho^2}\cos^2\tau}}\left(arctanh\frac{w'(Y)}{\sqrt{c}\sqrt{1+\frac{4K_N}{c\rho^2}\cos^2\tau}} - a_0\right)$$

$$e^{F(S)} = \frac{\cos^2\tau}{K_N}(1+\frac{4K_N}{c\rho^2})\frac{1}{\cosh\left(\sqrt{c(1+\frac{4K_N}{c\rho^2}\cos^2\tau)}(S-S_0)+\Sigma_0\right)}$$

where

$$\tanh\omega = \frac{1}{\sqrt{m}}sn(\sqrt{Cm}(Y - Y_0) + v_0|\frac{1}{m})\ldots$$

Recover example in JDG

$$R = x + a \int_0^y \sin^2 S(t) dt,$$

$$S = \cot^{-1} z(y), y \in [0, \frac{\pi}{2}]$$

$$\int_0^{z(y)} \frac{dz}{\sqrt{z^4 + c^2 z^2 + b^2}} = \frac{\pi}{2} - y, c^2 = 1 + b^2 + a^4, y \in [0, \frac{\pi}{2}]$$

Recover example in JDG

$$\int_0^\infty \frac{dz}{\sqrt{z^4 + c^2 z^2 + b^2}} = \frac{\pi}{2}$$

$$a^2 \int_0^\infty \frac{dz}{(1+z^2)\sqrt{z^4 + c^2 z^2 + b^2}} = \frac{\beta}{2}$$

and extend the functions R, S by symmetry in the $y = \frac{\pi}{2}$ axis...
Recover examples: Shi-Tam-Wan, Wolf, Li-Tam

Pseudo-Riemann surfaces

Mapping between surfaces: Isothermal coordinates...

$$g = e^{f(v,w)} dvdw = e^{f(x,y)}(dx^2 - \epsilon^2 dy^2),$$

where $v = x + \epsilon y, w = x - \epsilon y, \epsilon = 1, i$

$$h = e^{F(V,W)} dVdW = e^{F(R,S)}(dR^2 - \delta^2 dS^2),$$

where
$v = x + \epsilon y, w = x - \epsilon y, V = R + \delta S, W = R - \delta S, \delta = 1, i$

Harmonic maps

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M, N pseudo-Riemann surfaces, $u\colon M \to N$
In isothermal coordinates:

$$E(u) = \frac{1}{2} \int (R_x^2 - \frac{1}{\epsilon^2} R_y^2 - \delta^2 S_x^2 + \frac{\delta^2}{\epsilon^2} S_y^2) dxdy$$

Euler-Lagrange equations of the energy functional are a nonlinear system of elliptic or hyperbolic PDEs of second order.
Solutions: Harmonic (or Wave) Maps

Notation

$$V_v = \frac{1}{2}(V_x + \frac{1}{\epsilon}V_y) = \frac{1}{2}(R_x + \frac{1}{\epsilon}R_y + \delta S_x + \frac{\delta}{\epsilon}S_y),$$
$$V_w = \frac{1}{2}(V_x - \frac{1}{\epsilon}V_y) = \frac{1}{2}(R_x - \frac{1}{\epsilon}R_y + \delta S_x - \frac{\delta}{\epsilon}S_y),$$
$$W_v = \frac{1}{2}(W_x + \frac{1}{\epsilon}W_y) = \frac{1}{2}(R_x + \frac{1}{\epsilon}R_y - \delta S_x - \frac{\delta}{\epsilon}S_y),$$
$$W_w = \frac{1}{2}(W_x - \frac{1}{\epsilon}W_y) = \frac{1}{2}(R_x - \frac{1}{\epsilon}R_y - \delta S_x + \frac{\delta}{\epsilon}S_y).$$

Harmonic maps

A map $u = (R, S)$ is harmonic if it satisfies the system

$$V_{vw} + F_V(V, W)\, V_v V_w = 0$$

$$W_{vw} + F_W(V, W)\, W_v W_w = 0,$$

where $V = R + \delta S$, $W = R - \delta S$.

Integrability

$u = R + iS$ harmonic iff

$$\left(e^{F(V,W)} V_v W_v\right)_w = 0$$

and

$$\left(e^{F(V,W)} V_w W_w\right)_v = 0$$

Consider locally a conformal change of coordinates s.t.

$$\left(e^{F(V,W)} V_v W_v\right) = 1, \ \left(e^{F(V,W)} V_w W_w\right) = 1$$

Change of coordinates

If

$$\left(e^{F(V,W)}V_v W_v\right) = \Lambda(v), \ \left(e^{F(V,W)}V_w W_w\right) = M(w)$$

then

$$\zeta = \int \frac{1}{\sqrt{\Lambda(v)}}\, dv, \quad \eta = \int \frac{1}{\sqrt{M(w)}}\, dw$$

Parametrization

Harmonic maps equations are equivallent to...

$$R_x R_y - \delta^2 S_x S_y = 0$$

and

$$\left(R_x^2 + \frac{1}{\epsilon^2} R_y^2\right) - \delta^2 \left(S_x^2 + \frac{1}{\epsilon^2} S_y^2\right) = 4e^{-F(R,S)}.$$

Parametrization

$$
\begin{aligned}
R_x &= 2e^{-\frac{F}{2}} \cosh \Omega \cosh \Theta \\
R_y &= 2\,\epsilon\, e^{-\frac{F}{2}} \sinh \Omega \sinh \Theta \\
S_x &= \frac{2}{\delta} e^{-\frac{F}{2}} \cosh \Omega \sinh \Theta \\
S_y &= \frac{2\epsilon}{\delta} e^{-\frac{F}{2}} \sinh \Omega \cosh \Theta,
\end{aligned}
$$

where

$$\Omega = \frac{\delta}{\epsilon}\omega, \quad \Theta = \delta\theta$$

Compatability conditions

Since $V_{vw} = V_{wv}$ and $W_{vw} = W_{wv}$ we get:

$$\Omega_w + \Theta_w = \frac{F_w}{2} - F_V V_w,$$
$$-\Omega_v + \Theta_v = \frac{F_v}{2} - F_V V_v.$$

sine-Gordon type equation

Since $\partial^2_{xy}\Theta = \partial^2_{yx}\Theta$ we get:

$$\Omega_{xx} - \epsilon^2 \Omega_{yy} = -2K_N \sinh 2\Omega.$$

where

$$K_N = -2F_{VW}e^{-F} = -\frac{1}{2}\left(F_{RR} - \delta^2 F_{SS}\right)\, e^{-F(R,S)},$$

and $\Omega = \frac{\delta}{\epsilon}\omega$

Observation

solutions of Beltrami equations

$$\frac{V_w}{V_v} = e^{-2\Omega},$$

$$\frac{W_w}{W_v} = e^{2\Omega},$$

Bäcklund transformation

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$\Omega_{vw} = -\frac{K_N}{2}\sinh(2\Omega)$, Bäcklund transformation
$2\Theta_{vw} = e^{-F}\left(\left(F_V^2 - F_{VV}\right)e^{2\Theta} - \left(F_W^2 - F_{WW}\right)e^{-2\Theta}\right)$

Bäcklund transformation

Let

$$e^F = \frac{1}{(aR + \delta^2 bS)^2}.$$

If $\delta = \epsilon = 1, a = 1, b = 0$ then auto-Bäcklund of

$$\omega_{xx} - \omega_{yy} = 2\sinh(2\omega),$$

If $\delta = i, \epsilon = 1, a = 1, b = 0$ then auto-Bäcklund of

$$\omega_{xx} - \omega_{yy} = 2\sin(2\omega).$$

Constant Curvature: Uniform Approach

We have to solve

$$\Omega_{xx} + \frac{1}{\epsilon^2}\Omega_{yy} = -\frac{K_N}{2}\sinh 2\Omega, \text{ where } \Omega = \frac{\delta}{\epsilon}\omega.$$

and Beltrami equations

$$\frac{V_w}{V_v} = e^{-2\Omega},$$

$$\frac{W_w}{W_v} = e^{2\Omega},$$

Assume

$$\omega = \omega(ax + by)$$

one soliton solution...

Interesting Problems

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1. Find new solutions to the sinh-Gordon equation and the corresponding harmonic maps.
2. The generalized sine-Gordon equation is interesting on its own right. Construct new families of solutions.
3. Multidimensional cases (under current investigation)

Publications

- Beltrami equation for the harmonic diffeomorphisms between surfaces. A. Fotiadis and C Daskaloyannis, Nonlinear Analysis (2022).
- Unified formalism for harmonic maps between Riemann or Lorentz surfaces. A. Fotiadis and C Daskaloyannis, Accepted in Journal of Geometry and Physics
- New examples of harmonic maps to the hyperbolic plane via Backlund transformation. G. Polychrou, E. Papageorgiou, A. Fotiadis, C. Daskaloyannis. Revista Matematica Complutense (2023)

Summary

- Harmonic maps between riemannian surfaces
- Generalization
- Open problems

Thank you very much for your attention!