The affine-additive group
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1 Definition-group structure

Definition-group structure

- Let $\mathrm{SU}(n,1)$ be (the triple cover) of the group of holomorphic isometries of the complex hyperbolic space $\mathbf{H}^n_{\mathbb{C}}$. The latter is realised by the Siegel domain

$$\mathbf{H}^n_{\mathbb{C}} = \{\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n \mid 2\Re(\zeta_1) + \sum_{i=2}^{n} |\zeta_i|^2 < 0\}.$$

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- We are in particular interested in two types of elements of $\mathrm{SU}(n,1)$.

Definition-group structure

- *Dilations* $D_\lambda,\ \lambda > 0$. Those correspond to elements of $\mathrm{SU}(n,1)$ of the following form

$$D(\lambda) = \begin{pmatrix} \sqrt{\lambda} & 0 & 0 \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1/\sqrt{\lambda} \end{pmatrix}.$$

The group D comprising elements of the above form is abelian and isomorphic to $\mathbb{R}_{>0},\cdot)$.

Definition-group structure

- *Heisenberg translations* $N(\zeta, t)$. Those correspond to elements of $\mathrm{SU}(n,1)$ of the following form:

$$N(z,t) = \begin{pmatrix} 1 & -\sqrt{2}\bar{z} & -\sum_{i=1}^{n} |z_i|^2 + it \\ 0 & I_{n-2} & \sqrt{2}z^t \\ 0 & 0 & 1 \end{pmatrix},$$

where $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $z_i = x_i + iy_i$, $t \in \mathbb{R}$.

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  where $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $z_i = x_i + iy_i$, $t \in \mathbb{R}$.
- The group N comprising Heisenberg translations is isomorphic to the $(n-1)$-*th Heisenberg group* $\mathbb{H}^{n-1}$ for $n > 1$.

Definition-group structure

- If $n = 1$, then

$$N = N(t) = \begin{pmatrix} 1 & it \\ 0 & 1 \end{pmatrix}, \quad t \in \mathbb{R},$$

and the group N is isomorphic to $(\mathbb{R}, +)$.

Definition-group structure

- We now consider matrices of the form

$$S(z,t,\lambda) = N(z,t)D(\lambda) = \begin{pmatrix} \sqrt{\lambda} & -\sqrt{2}\bar{z} & \frac{-\sum_{i=1^n}|z_i|^2+it}{\sqrt{\lambda}} \\ 0 & I_{n-2} & \frac{\sqrt{2}z^t}{\sqrt{\lambda}} \\ 0 & 0 & \frac{1}{\sqrt{\lambda}}. \end{pmatrix}$$

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- Via this set of matrices we find a 1–1 and onto correspondence of the set $\mathbb{R}_{>0} \times \mathbb{H}^{n-1}$ with the complex hyperbolic space $\mathbf{H}^n_{\mathbb{C}}$.

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- In fact, the map is

$$(\lambda, z, t) \mapsto \left(-\sum_{i=1}^{n-1} |z_i|^2 - \lambda + it,\ \sqrt{2}z \right)$$

and it just desribes the foliation of $\mathbf{H}^n_{\mathbb{C}}$ by $2n-1$-horospheres which are all copies of the Heisenberg group $\mathbb{H}^{n-1}$.

Definition-group structure

- If $S(\lambda',z',t')$ and $S(\lambda,z,t)$ are two such matrices, from the matrix multiplication

$$S(\lambda',z',t')S(\lambda,z,t)$$

we obtain the following group multiplication for the set $\mathbb{R}_{>0}\times\mathbb{H}^{n-1}$:

$$(\lambda',z',t')*(\lambda,z,t)=\left(\lambda'\lambda,\ z'+\sqrt{\lambda'}z,\ t'+t\lambda'+2\sqrt{\lambda'}\Im(z'\cdot\bar{z})\right). \tag{1.1}$$

Definition-group structure

- We next consider the set

$$\mathcal{AA}^n = \mathbb{R} \times (\mathbb{R}_{>0} \times \mathbb{H}^{n-1})$$

with coordinates (a, λ, z, t) and multiplication law

$$p' * p = \left(a' + a,\ \lambda'\lambda,\ z' + \sqrt{\lambda'}z,\ t' + t\lambda' + 2\sqrt{\lambda'}\Im(z' \cdot \bar{z})\right), \tag{1.2}$$

for each $p = (a, \lambda, z, t), p' = (a', \lambda', z', t')$.

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for each $p = (a, \lambda, z, t)$, $p' = (a', \lambda', z', t')$.

- Then the set $\mathcal{AA}^n$ with the multiplication law $*$ as above is a group with neutral element $e = (0, 1, 0_n, 0)$ and such that for each $p = (a, \lambda, z, t) \in \mathcal{AA}^n$ its inverse is

$$p^{-1} = (-a,\ 1/\lambda,\ -\sqrt{\lambda}z,\ -t/\lambda).$$

Definition-group structure

Definition

We shall call the group $(\mathcal{AA}^n, *)$ *the (n-th) affine-additive group.*

Lie group structure

- We write as above $p' = (a', \lambda', z', t;)$ and $p = (a, \lambda, z, t)$. From the differential of the left translation $L_{p'}(p) = p' * p$, we obtain the folowing invariant basis of the tangent bundle of $\mathcal{A}\mathcal{A}^n$:

$$X_i = \sqrt{\lambda}(\partial_{x_i} + 2y_i\partial_t), \qquad Y_i = \sqrt{\lambda}(\partial_{y_i} - 2x_i\partial_t),$$
$$V = 2\lambda\partial_\lambda, \quad U = \partial_a + 2\lambda\partial_t, \quad W = -\partial_a. \tag{1.3}$$

Lie group structure

- The only non vanishing Lie brackets are the following

$$[X_i, Y_i] = [U, V] = -2(V + W),$$
$$[X_i, V] = -X_i, \quad [Y_i, V] = Y_i. \tag{1.4}$$

Contact form

- We now consider the 1–form

$$\theta = \frac{dt + 2\sum_{i=1}^{n-1}(x_i dy_i - y_i dx_i)}{2\lambda} - da. \tag{1.5}$$

Contact form

Proposition

The pair $(\mathcal{AA}^n, \theta)$ *is a* $(2n+1)$*-dimensional contact manifold. In fact, the following hold:*

a) *The form* θ *is left–invariant.*

b) $\theta \wedge (d\theta)^n \neq 0$.

c) $\theta(W) = 1$ *and* $\langle W \rangle = \ker(d\theta)^n$.

d) $\ker\theta = \langle X_i, Y_i, U, V \rangle$.