New examples of harmonic maps to the hyperbolic plane via Bäcklund transformation

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Introduction

1. Sine-Gordon equations and applications to Geometric Analysis.
2. Harmonic maps and the elliptic sinh-Gordon.
3. Families of solutions of the elliptic sinh-Gordon.
4. Bäcklund transformation connecting the elliptic sinh-Gordon with the elliptic sine-Gordon.
5. Construction of harmonic maps from a surface to a hyperbolic plane via Bäcklund transformation.

Sinh-Gordon type equations

(Hyperbolic) Sine-Gordon

$$w_{xx} - w_{yy} = \sin(2w) \Leftrightarrow \Box w = \sin(2w).$$

Hyperbolic Sinh-Gordon

$$w_{xx} - w_{yy} = \sinh(2w) \Leftrightarrow \Box w = \sinh(2w).$$

Elliptic Sine-Gordon

$$w_{xx} + w_{yy} = \sin(2w) \Leftrightarrow \Delta w = \sin(2w).$$

Elliptic Sinh-Gordon

$$w_{xx} + w_{yy} = \sinh(2w) \Leftrightarrow \Delta w = \sinh(2w).$$

Sinh-Gordon type equations

Generalised sinh-Gordon

$$\Box_\delta w = \frac{2a}{\kappa} \sinh\left(2\kappa w\right),$$

where $\Box_\delta = \frac{\partial^2}{\partial x^2} - \delta^2 \frac{\partial^2}{\partial y^2}$, with $\delta, \kappa \in \{1, i\}$ and $a \in \mathbb{R}$.

δ	κ	Equation
1	1	$w_{xx} - w_{yy} = 2a \sinh(2w)$
1	i	$w_{xx} - w_{yy} = 2a \sin(2w)$
i	1	$w_{xx} + w_{yy} = 2a \sinh(2w)$
i	i	$w_{xx} + w_{yy} = 2a \sin(2w)$

Sine-Gordon equations and Geometry

Applications

1. Surfaces of constant negative curvature in $\mathbb{R}^3$.
2. Hopf's conjecture and Wente Torus.
3. Hypersurfaces of constant sectional curvature.
4. Harmonic diffeomorphisms between Riemann surfaces.
5. (Lorentz)-Wave maps.

CMC surfaces and Sinh-Gordon equation

Theorem, Kenmotsu

Let a surface $\Sigma : D \to \mathbb{R}^3$, where $D \subset \mathbb{R}^2$, be of constant and non-zero mean curvature H and let $p \in D$ be a non-umbilic point of Σ. (That is, the two principal curvatures κ_1 and κ_2 are different from each other at each point of the surface.) Then there exists isothermal coordinates (x, y) in a neighborhood $U(p) \subset D$:

$$I_\Sigma = \frac{e^{2w}}{2H}(dx^2 + dy^2), \quad II_\Sigma = e^w \cosh w dx^2 + e^w \sinh w dy^2,$$

where $w = w(x, y)$ satisfies the elliptic sinh-Gordon equation

$$\Delta w + H \sinh(2w) = 0.$$

Conversely, for a given positive number H and a solution w of the elliptic sinh-Gordon, there exists a CMC surface uniquely up to isometries of $\mathbb{R}^3$ with fundamental forms as above.

Harmonic maps

Differential of a map

Let (M^m, g) and (N^n, h) smooth Riemannian manifolds equipped with local coordinates $(x^1, x^2, ..., x^m)$ and $(y^1, y^2, ..., y^n)$ and a C^∞ map $u : M \to N$. We define the differential of u, du and its Hilbert-Schmidt norm in local coordinates

$$|du|^2 = g^{ij} h_{\alpha\beta}(u) \left(\frac{\partial u^\alpha}{\partial x^i}\right) \left(\frac{\partial u^\beta}{\partial x^j}\right).$$

Energy density

Let $u : (M, g) \to (N, h)$ a C^∞ map. We define the energy density as

$$e(u)(x) = \frac{1}{2}|du|^2(x), \quad x \in M.$$

Harmonic maps

Energy of a map

Let (M, g) be compact. We define the energy of a map, the Dirichlet integral, as

$$E(u) = \int_M e(u) dM.$$

Harmonic maps

A map $u : (M, g) \to (N, h)$ is called harmonic if it is a critical point of the energy integral.

Euler-Lagrange equations-Tension field

$$\tau(u)^{\gamma}_{ij} = \nabla(du)^{\gamma}_{ij} = \frac{\partial^2 u^{\gamma}}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial u^{\gamma}}{\partial x^k} + \frac{\partial u^{\alpha}}{\partial x^i} \frac{\partial u^{\beta}}{\partial x^j} \tilde{\Gamma}^{\gamma}_{\alpha\beta}(u) = 0$$

Harmonic maps

Examples of harmonic maps

1. Constant maps and identity maps.
2. Isometries are harmonic maps. Note that the composition of a harmonic map with an isometry is harmonic.
3. Minimal submanifolds in $\mathbb{R}^n$ have harmonic Gauss maps.
4. A smooth map $\phi : A \to \mathbb{R}^n$, $A \subset \mathbb{R}^m$ is harmonic if-f each component is a harmonic function.
5. A smooth map $\phi : (M, g) \to \mathbb{R}^n$ is harmonic if-f each of its components is a harmonic function on (M, g).
6. Let $\phi : (M, g) \to (N, h)$ harmonic map and $f : (N, h) \to (P, k)$ totally geodesic then $f \circ \phi$ is harmonic.
7. The holomorphic maps between Kähler manifolds are harmonic.

Harmonic maps between Riemann surfaces

Isothermal coordinates

Let $u : M \to N$ be a map between Riemann surfaces (M, g), (N, h). The map u is locally represented by $u = u(z, \bar{z}) = R + iS$, where $z = x + iy$. From now on, we use the standard notation:

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

Consider an isothermal coordinate system (x, y) on M such that $g = e^{f(z,\bar{z})}|dz|^2$, where $z = x + iy$, and an isothermal coordinate system (R, S) on N such that $h = e^{F(u,\bar{u})}|du|^2$, where $u = R + iS$. The Gauss curvature on the target is given by the formula

$$K_N(u, \bar{u}) = -\frac{1}{2}\Delta F(u, \bar{u})e^{-F(u,\bar{u})}.$$

Harmonic maps between Riemann surfaces

Harmonic map equation

In isothermal coordinates, a map between two surfaces is harmonic if and only if it satisfies

$$\partial_{z\bar{z}} u + \partial_u F(u, \bar{u}) \partial_z u \partial_{\bar{z}} u = 0.$$

Note that this equation only depends on the conformal structure of N.

Proposition

If $\kappa : \Sigma_1 \to \Sigma_2$ is a holomorphic or antiholomorphic map between Riemann surfaces and $f : \Sigma_2 \to N$ harmonic then so is $f \circ \kappa$.

Harmonic maps between Riemann surfaces

Hopf differential

The Hopf differential of u is given by

$$\Lambda(z)dz^2 = \left(e^{F(u,\bar{u})}\partial_z u \partial_z \bar{u}\right) dz^2.$$

We may assume that Λ does not vanish locally.

Propostion, Hopf

A necessary and sufficient condition for a C^2 map u with non vanishing Hopf differential, with almost everywhere non vanishing Jacobian, to be a harmonic map, is that

$$e^{F(u,\bar{u})}\partial_z u \partial_z \bar{u} = e^{-\mu(z)},$$

where $\mu(z)$ is a holomorphic function.

Harmonic maps between Riemann surfaces

Theorem (Existence of harmonic maps between surfaces)

Let Σ_1 and Σ_2 are compact surfaces, $\partial\Sigma_2 = \varnothing$ and $\pi_2(\Sigma_2) = 0$. If $\phi : \Sigma_1 \to \Sigma_2$ is a continuous map with finite energy, then there exists a harmonic map $u : \Sigma_1 \to \Sigma_2$ which is homotopic to ϕ, coincides with ϕ on $\partial\Sigma_1$ in case $\partial\Sigma_1 \neq \varnothing$ and is energy minimizing among all such maps.

Lamaire (J. Diff. Geom., 1978, Ann. Sc. Norm. Sup. Pisa, 1982).
Sacks-Unlenbeck, Case $\partial\Sigma_1 = \varnothing$ (Ann. Math., 1981).
Schoen-Yau (Ann. Math., 1979)
Jost (M. Z., 1983)

Harmonic maps between Riemann surfaces

Propostion, Minksy (J. Diff. Geom.), Wolf (J. Diff. Geom.)

Let $u : M \to N$ be a harmonic map. Then, it satisfies the Beltrami equation

$$\frac{\partial_{\bar{z}} u}{\partial_z u} = e^{-2w+i\phi},$$

and ϕ is a harmonic function, i.e. $\partial^2_{z\bar{z}}\phi = 0$. Furthermore, if ψ is the conjugate harmonic function to ϕ, then

$$K_N = -\frac{2\partial^2_{z\bar{z}} w}{\sinh 2w} e^{\psi},$$

where K_N is the curvature of the target manifold N.

Propostion, Fotiadis-Daskaloyannis (Nonlinear Analysis)

Let $w = w(x, y)$ be a solution of the sinh-Gordon equation

$$\Delta w = 2\sinh(2w)$$

where $\Delta = \partial^2_{xx} + \partial^2_{yy}$ is the Laplacian with the flat metric and let $u = u(z, \bar{z})$ be a solution of the Beltrami equation

$$\frac{\partial_{\bar{z}} u}{\partial_z u} = e^{-2w},$$

and $z = x + iy$ lie in an open simply connected subset Ω of $\mathbb{C}$ where the map u is a well defined C^2 map. Without loss of generality we assume that Ω contains the origin. Then, u is a harmonic map, if the curvature of the target is -1.

Propostion, Fotiadis-Daskaloyannis (Nonlinear Analysis)

Let $\Phi(x, y) = constant$ be the solution of the characteristics

$$\frac{dy}{dx} = i \coth(w(x, y)),$$

where $w(x, y)$ is a solution of the elliptic sinh-Gordon $\Delta w = 2 \sinh(2w)$. Then, a harmonic map that corresponds to w is of the form

$$u(x, y) = Re\Phi(x, y) + iIm\Phi(x, y) = R(x, y) + iS(x, y).$$

Bäcklund transformation

Proposition, Fotiadis-Daskaloyannis (Nonlinear Analysis)

The system

$$\partial_x w - \partial_y \theta = -2 \sinh w \sin \theta,$$
$$\partial_y w + \partial_x \theta = -2 \cosh w \cos \theta,$$

is a *Bäcklund transformation* that connects a solution $w = w(x, y)$ of the elliptic sinh-Gordon equation and a solution $\theta = \theta(x, y)$ of the sine-Gordon equation

$$\Delta \theta = -2 \sin(2\theta).$$

Definition

We say that the pair of functions (w, θ) is in the class (BT), if the functions w and θ satisfy the system above.

Elliptic functions

Elliptic Integral of first kind

$$u = \int_0^{\phi} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}, \quad 0 < k < 1, \quad \phi = am(u, k)$$

Jacobi elliptic functions

$$sn(u, k) = \sin(\phi), \quad \left(\frac{dy}{dx}\right)^2 = (1 - y^2)(1 - k^2 y^2), y(0) = 0$$

$$cn(u, k) = \cos(\phi), \quad \left(\frac{dy}{dx}\right)^2 = (1 - y^2)(1 - k^2 + k^2 y^2), y(0) = 1$$

$$dn(u, k) = \sqrt{1 - \sin^2(\phi)}, \quad \left(\frac{dy}{dx}\right)^2 = (y^2 - 1)(1 - k^2 - y^2), y(0) = 1$$

The elliptic sinh-Gordon equation

Tanh-method

If w is a solution of the sinh-Gordon equation of the form

$$w(x,y) = 2\,\text{arctanh}(F(x)G(y)),$$

then the functions F, G satisfy the differential equations

$$(F'(x))^2 = AF^4(x) + BF^2(x) + C$$
$$(G'(y))^2 = -CG^4(y) - (B-4)G^2(y) - A,$$

where A, B, C are arbitrary constants.

The elliptic sinh-Gordon equation

Proposition, Kenmotsu

Let $w_0 > 0$, and $\alpha, \beta > 0$, such that $\alpha + \beta = \cosh w_0 > 1$. Consider $f(x)$, $g(y)$ such that

$$(f')^2 = f^4 - 4\left(1 + \alpha^2 - \beta^2\right) f^2 + 4^2\alpha^2$$
$$(g')^2 = g^4 - 4\left(1 + \beta^2 - \alpha^2\right) g^2 + 4^2\beta^2,$$

with $f(0) = 0$, $f'(0) = -4\alpha$, $g(0) = 0$, $g'(0) = -4\beta$. Then the function $w(x, y)$ given by

$$\tanh \frac{w(x, y)}{2} = \tanh \frac{w_0}{2} e^{-\int_0^x f(t)dt} e^{-\int_0^y g(s)ds},$$

is such that $\Delta w = 2\sinh(2w)$ and $w(0, 0) = w_0$.

The elliptic sinh-Gordon equation

Definition of (BT_0) Class

If (w, θ) satisfy the Bäcklund transformation with initial data $w_y(x, 0) = 0$ and $\theta(0, 0) = \frac{\pi}{2}$, then we say that (w, θ) belong in (BT_0) class.

$$X = X(x) = \int_0^x \cosh w(t, 0)dt = \int_0^x \frac{1 + F^2(t)G^2(0)}{1 - F^2(t)G^2(0)}dt,$$

$$Y = Y(x, y) = \int_0^y \sinh w(x, s)ds = \int_0^y \frac{2F(x)G(s)}{1 - F^2(x)G^2(s)}ds.$$

The elliptic sinh-Gordon equation

Tanh-method and Bäcklund transformation, P., Papageorgiou, Fotiadis, Daskaloyannis (Rev. Mat. Comp.)

If $w(x,y) = 2\,\text{arctanh}(F(x)G(y))$ a solution of the elliptic sinh-Gordon and let $b(x) = \frac{F'(x)}{2F(x)}$ and $(w,\theta) \in (BT_0)$, then

$$\tan\frac{\theta(x,y)}{2} = \frac{1}{b(x)}\left(\sqrt{b^2(x)-1}\,\tan(J_1(x)+J_2(x,y)) - 1\right),$$

is a family of solution of the elliptic sine-Gordon, where

$$J_1 = J_1(x) = \arctan\left(\frac{b(x)+1}{\sqrt{b^2(x)-1}}\right),$$

$$J_2 = J_2(x,y) = \sqrt{b^2(x)-1}\;Y(x,y).$$

The elliptic sinh-Gordon equation

Proposition P. (J. Elliptic and Parabolic Equ)

If w is a solution of the elliptic sinh-Gordon equation of the form

$$\sinh w(x,y) = \tan(A(x) + B(y)),$$

then the functions $a(x)$ and $b(y)$ satisfy the differential equations

$$(a'(x))^2 = -(a(x))^4 + c_1(a(x))^2 + c_2,$$

$$(b'(y))^2 = -(b(y))^4 + (8 - c_1)(b(y))^2 + c_3,$$

where $a(x) = A'(x)$, $b(y) = B'(y)$ and c_1, c_2 and c_3 are constants such that $4c_1 = 16 + c_3 - c_2$.

Corollary P. (J. Elliptic and Parabolic Equ)

By using the initial conditions $A'(0) = a(0) = B'(0) = b(0) = 0$, we have $c_2 = a'(0)^2 = 16\alpha^2$, $c_3 = b'(0)^2 = 16\beta^2$ and $c_1 = 4(1 - \alpha^2 + \beta^2)$ $(\alpha, \beta \in \mathbb{R})$ and the equations of the previous slide turn into

$$(a'(x))^2 = -(a(x))^4 + 4(1 - \alpha^2 + \beta^2)(a(x))^2 + 16\alpha^2,$$

$$(b'(y))^2 = -(b(y))^4 + 4(1 + \alpha^2 - \beta^2)(b(y))^2 + 16\beta^2.$$

Example

Let a solution of the elliptic sinh-Gordon

$$\sinh w(x,y) = \frac{\sinh(2x) + \sinh(2y)}{1 - \sinh(2x)\sinh(2y)}.$$

We have to solve the following equation to construct the corresponding harmonic map:

$$\frac{dy}{dx} = i \coth w(x,y) = i\,\frac{\cosh(2x)\cosh(2y)}{\sinh(2x) + \sinh(2y)}.$$

Example

By calculations we find that a corresponding harmonic map is

$$u(x,y) = R(x,y) + iS(x,y),$$

$$R(x,y) = sech(2y) - \sinh(2x)\tanh(2y),$$

$$S(x,y) = \sinh(2x)sech(2y) + \tanh(2y) - 2y.$$

An implicit formula for the metric on the target of curvature -1 is

$$\frac{e^{F(u,\bar{u})}}{4} du d\bar{u} = \frac{\cosh^2(2x)\cosh^2(2y)dx^2 + (\sinh(2x) + \sinh(2y))^2 dy^2}{(1 - \sinh(2x)\sinh(2y))^2}$$

The elliptic sinh-Gordon equation

Proposition P. (J. Elliptic and Parabolic Equ)

If $\theta(x, y)$ is a solution of the elliptic sine-Gordon equation of the form

$$\theta(x, y) = 2 \arctan(F(x)G(y)),$$

then the functions $F(x)$ and $G(y)$ satisfy the following equations

$$F'(x)^2 = AF^4(x) + BF^2(x) + C,$$

$$G'(y)^2 = CG^4(y) - (4 + B)G^2(y) + A,$$

where A, B and C are arbitrary constants.

The elliptic sinh-Gordon equation

$$Y = Y(y) = \int_0^y \cos\theta(0,s)ds, \quad X = X(x,y) = \int_0^x \sin\theta(t,y)dt.$$

Proposition P. (J. Elliptic and Parabolic Equ)

If $\theta(x,y)$ is a solution of the elliptic sine-Gordon equation of the form $\theta(x,y) = 2\arctan(F(x)G(y))$ with initial conditions $F'(0) = 0$ and $w(0,0) = 0$ then

$$\tanh(\frac{w(x,y)}{2}) = \frac{2 - K\tan(Y) + \sqrt{K^2+4}\tanh(\frac{\sqrt{K^2+4}}{2}X)}{\sqrt{K^2+4} + (2 - K\tan(Y))\tanh(\frac{\sqrt{K^2+4}}{2}X)},$$

is a family of solutions of the elliptic sinh-Gorodon, where $K = K(y) = \frac{H'(y)}{H(y)}$ and $H(y) = \frac{1}{G(y)}$.

The elliptic sinh-Gordon equation

Proposition P. (J. Elliptic and Parabolic Equ)

If θ is a solution of the elliptic sine-Gordon equation of the form

$$\sin\theta(x,y) = \tanh(C(x) + D(y))$$

then the functions $c(x)$ and $d(y)$ satisfy the differential equations

$$(c'(x))^2 = (c(x))^4 + c_4(c(x))^2 + c_5,$$

$$(d'(y))^2 = (d(y))^4 - (8 + c_4)(d(y))^2 + c_6,$$

where $c(x) = C'(x)$, $d(y) = D'(y)$ and c_4, c_5, c_6 are arbitrary constants such that $16 + 4c_4 = c_6 - c_5$.

The elliptic sinh-Gordon equation

$$X(x) = \int_0^x \sin\theta(t,0)dt, \quad Y(x,y) = \int_0^y \cos\theta(x,s)ds.$$

Proposition P. (J. Elliptic and Parabolic Equ)

If $\theta(x,y)$ is a solution of the elliptic sine-Gordon of the form $\sin\theta(x,y) = \tanh(C(x) + D(y))$ and $d(0) = 0$ then

$$\tanh\left(\frac{w(x,y)}{2}\right) = L\frac{\tanh(\frac{w(0,0)}{2})e^{-2X} + L\tan(\frac{\sqrt{|4-c^2(x)|}}{2}Y)}{L - \tanh(\frac{w(0,0)}{2})e^{-2X}\tan(\frac{\sqrt{|4-c^2(x)|}}{2}Y)},$$

where $L = L(x) = \frac{\sqrt{|4-c(x)^2|}}{c(x)-2}$, is a solution of the elliptic sinh-Gordon.

Construction of harmonic maps

Lemma P., Papageorgiou, Fotiadis, Daskaloyannis (Rev. Mat. Comp.)

Assume that $(w, \theta) \in (BT)$. Then, there exist functions R and S such that

$$\begin{aligned} \partial_x S &= 2S \cosh w \sin \theta, \\ \partial_y S &= 2S \sinh w \cos \theta, \\ \partial_x R &= 2S \cosh w \cos \theta, \\ \partial_y R &= -2S \sinh w \sin \theta. \end{aligned}$$

Let (R, S) satisfy the system above. Then,

$$u(x, y) = R(x, y) + iS(x, y)$$

is the harmonic map to the hyperbolic plane that corresponds to w.

Construction of harmonic maps

$$
\begin{aligned}
I_1 = I_1(x) &= \int_0^x \cosh w(t,0)\sin\theta(t,0)dt,\\
I_2 = I_2(x,y) &= \int_0^y \sinh w(x,s)\cos\theta(x,s)ds,\\
I_3 = I_3(x) &= \int_0^x e^{2I_1(t)}\cosh w(t,0)\cos\theta(t,0)dt,\\
I_4 = I_4(x,y) &= e^{2I_1(x)}\int_0^y e^{2I_2(x,s)}\sinh w(x,s)\sin\theta(x,s)ds.
\end{aligned}
$$

Proposition P., Papageorgiou, Fotiadis, Daskaloyannis (Rev. Mat. Comp.)

Define the function S by

$$S(x,y) = S(0,0)e^{2(I_1+I_2)},$$

and the function R by

$$R(x,y) = R(0,0) + 2S(0,0)(I_3 - I_4).$$

Then,

$$u(x,y) = R(x,y) + iS(x,y)$$

is the harmonic map that corresponds to w. The domain of R and S is the largest possible open simply connected subset of $\mathbb{C}$ containing the origin so that the above expressions make sense.

Construction of harmonic maps

Example 1

Suppose that $\tanh\left(\frac{w(x,y)}{2}\right) = \frac{2y}{\cosh 2x}$ a solution of the elliptic sinh-Gordon and consider the initial data $R(0,0) = 0$, $S(0,0) = -\frac{1}{4}$. Using the Bäcklund transformation we find that $\tan\left(\frac{\theta(x,y)}{2}\right) = \coth x$ the corresponding solution of the elliptic sine-Gordon. Then,

$$
\begin{aligned}
I_1 &= \frac{1}{2}\log\cosh 2x, \\
I_2 &= \frac{1}{2}\log\left(1 - \frac{4y^2}{\cosh^2 2x}\right), \\
I_3 &= -x, \\
I_4 &= 2y^2\tanh 2x.
\end{aligned}
$$

Example 1

Then $u = R + iS$ where

$$R = \frac{x}{2} + y^2 \tanh 2x$$
$$S = \frac{y^2}{\cosh 2x} - \frac{\cosh 2x}{4}.$$

The domain of definition of u is the set

$$\Omega = \left\{(x, y) : \ |y| < \frac{1}{2}\cosh 2x\right\}$$

where the map is a well defined C^2 map whose Jacobian is almost everywhere non vanishing.

Construction of harmonic maps

Reminder

Family of solutions to elliptic sinh-Gordon

$$\tanh\left(\frac{w(x,y)}{2}\right) = F(x)G(y).$$

$$X = X(x) = \int_0^x \cosh w(t,0)dt = \int_0^x \frac{1+F^2(t)G^2(0)}{1-F^2(t)G^2(0)}dt,$$

$$Y = Y(x,y) = \int_0^y \sinh w(x,s)ds = \int_0^y \frac{2F(x)G(s)}{1-F^2(x)G^2(s)}ds.$$

Construction of harmonic maps

Theorem P., Papageorgiou, Fotiadis, Daskaloyannis (Rev. Mat. Comp.)

Assume that $w(x,y) = 2\,\mathrm{arctanh}(F(x)G(y))$, $b(x) = \frac{F'(x)}{2F(x)}$ and $(w,\theta) \in (BT_0)$. Define the functions S and R on the largest possible open simply connected subset of $\mathbb{C}$ containing the origin such that

$$S(x,y) = S(0,0)\frac{e^{2X(x)}(\sin\theta(x,y) + b(x))}{1+b(x)},$$

$$R(x,y) = R(0,0) + S(0,0)\frac{e^{2X(x)}\cos\theta(x,y)}{1+b(x)},$$

Then $u(x,y) = R(x,y) + iS(x,y)$ is the harmonic map that corresponds to w .

Example 2

Consider $S(0,0) = 1$, $R(0,0) = 0$ and $\tanh \frac{w(x,y)}{2} = \frac{\sqrt{2}}{2} \frac{\cosh(2\sqrt{2}y)}{\cos(2x)}$. We find

$$X = x + \operatorname{arctanh}(\tan(2x)), \quad b = b(x) = \tan(2x),$$

$$Y = \frac{1}{\sqrt{1 - \tan^2(2x)}} \operatorname{arctanh} \frac{\sinh(2\sqrt{2}y)}{\sqrt{1 - 2\sin^2(2x)}},$$

and

$$\tan \frac{\theta(x,y)}{2} = \frac{\sinh(2\sqrt{2}y) + \cos(2x) - \sin(2x)}{\cos(2x) - \sin(2x) - \sinh(2\sqrt{2}y)}.$$

Example 2

$$S(x,y) = e^{2x}\frac{1 + 2\cos(4x) - \cosh(4\sqrt{2}y)}{1 + \cosh(4\sqrt{2}y) - 2\sin(4x)}$$

and

$$R(x,y) = -4e^{2x}\frac{\cos(2x)\sinh(2\sqrt{2}y)}{1 + \cosh(4\sqrt{2}y) - 2\sin(4x)}.$$

Therefore, $u = R + iS$ is the harmonic map that corresponds to w. The domain of definition of u is the set

$$\Omega = \left\{(x,y) : \left|\frac{\sqrt{2}}{2}\frac{\cosh(2\sqrt{2}y)}{\cos(2x)}\right| < 1\right\}$$

where the map is a well defined C^2 map whose Jacobian is almost everywhere non vanishing.

Construction of harmonic maps

Second main result P., Papageorgiou, Fotiadis, Daskaloyannis (Rev. Mat. Comp.)

Let $\theta = \theta(x)$ be a solution to the sine-Gordon equation such that

$$\int_0^{\theta(x)} \frac{d\psi}{\sqrt{1 - \frac{4}{c^2}\sin^2(\psi)}} = c_1 + cx \Leftrightarrow \sin\theta(x) = \mathrm{sn}(cx + c_1 | \frac{4}{c^2}),$$

where $c, c_1 \in \mathbb{R}$. Then the associated solution w to the Bäcklund transformation is of the form

$$\tanh\frac{w(x,y)}{2} = F(x)G(y) = \frac{\sqrt{ab}\tan(\frac{\sqrt{ab}}{2}y + k)}{\theta'(x) - 2\cos\theta(x)},$$

where $a = 2\cos\theta(0) + \theta'(0)$, $b = 2\cos\theta(0) - \theta'(0)$, and $k \in \mathbb{C}$ such that $\tanh\frac{w(0,0)}{2} = -\sqrt{\frac{a}{b}}\tan(k)$.

Second main result P., Papageorgiou, Fotiadis, Daskaloyannis (Rev. Mat. Comp.)

Then, the harmonic map $u = R + iS$ that corresponds to w is

$$R(x,y) = R(0,0) + S(0,0)\cosh^2\frac{w(0,0)}{2} J\left(\frac{2\cos\theta(x) - \theta'(x)}{b}\right)$$

$$+ 2S(0,0)\frac{\sin\theta(x)(\cos(\sqrt{ab}y + 2k) - \cos(2k))}{2\cos\theta(0)\cos(2k) - \theta'(0)},$$

$$S(x,y) = S(0,0)\frac{2\cos\theta(x)\cos(\sqrt{ab}y + 2k) - \theta'(x)}{2\cos\theta(0)\cos(2k) - \theta'(x)},$$

where

$$J(t) = \int_1^t \frac{u^2 + \tanh^2\frac{w(0,0)}{2}}{u^2}\frac{bu^2 + a}{\sqrt{2(8 - ab)u^2 - a^2 - b^2u^4}}du.$$

Example 3

Consider the case when $\theta(0)=0$, $\theta'(0)=1$ and $\tanh\frac{w(0,0)}{2}=0$. Then, we compute

$$\sin\theta(x)=\mathrm{sn}(x|4),\quad \tanh\left(\frac{w(x,y)}{2}\right)=\frac{\sqrt{3}\tan\left(\frac{\sqrt{3}}{2}y\right)}{\theta'(x)-2\cos\theta(x)},$$

$$\begin{aligned}
I_1(x) &= I(x)=\frac{1}{2}\log(2\cos\theta(x)-\theta'(x)),\\
I_2(x,y) &= \frac{1}{2}\log\left(\frac{2\cos\theta(x)\cos(\sqrt{3}y)-\theta'(x)}{2\cos\theta(x)-\theta'(x)}\right),\\
I_3(x) &= \frac{1}{2}\int_1^{2\cos\theta(x)-\theta'(x)}\frac{u^2+3}{\sqrt{10u^2-u^4-9}}du,\\
I_4(x,y) &= \sin\theta(x)(\cos(\sqrt{3}y)-1).
\end{aligned}$$

Example 3

Let $S(0,0) = 1$ and $R(0,0) = 0$. Then, the corresponding harmonic map then is $u(x,y) = R(x,y) + iS(x,y)$, where

$$R(x,y) = J(2\cos\theta(x) - \theta'(x)) - 2\sin\theta(x)(\cos(\sqrt{3}y) - 1),$$

$$S(x,y) = 2\cos\theta(x)\cos(\sqrt{3}y) - \theta'(x),$$

where $J(t) = \int_1^t \frac{u^2+G^2(0)}{u^2}\frac{bu^2+a}{\sqrt{2(8-ab)u^2-a^2-b^2u^4}}du.$
The domain of definition of u is the set of

$$\left\{(x,y) : \left|\frac{\sqrt{3}\tan(\frac{\sqrt{3}}{2}y)}{\theta'(x) - 2\cos\theta(x)}\right| < 1\right\}$$

where the map is a well defined C^2 map whose Jacobian is almost everywhere non vanishing.

Example 4

Consider the solution of the elliptic sine-Gordon equation

$$\tan\left(\frac{\theta(x,y)}{2}\right) = 2y\sec(2x).$$

Using the Bäcklund transformation, we calculate the solution of the elliptic sinh-Gordon equation as

$$\tanh\left(\frac{w(x,y)}{2}\right) = \frac{\cos(y)(\sin(2x)-2y)+\sin(y)}{\cos(y)+(2y+\sin(2x))\sin(y)}.$$

Example 4

We can now construct the corresponding harmonic map

$$u(x,y) = R(x,y) + iS(x,y),$$

$$R(x,y) = \frac{\cos(2y)\cos^2(2x) + 4y\left(\sin(2x) + \sin(2y) - y\cos(2y)\right)}{4y^2 + \cos^2(2x)},$$

$$S(x,y) = 2x + \frac{4y\cos(2x)\cos(2y) - 2\cos(2x)(\sin(2x) + \sin(2y))}{4y^2 + \cos^2(2x)}.$$

The metric on the target in implicit form is

$$I = 4\frac{(3 + 8y^2 - \cos(4x) + 4\sin(2x)(\sin(2y) - 2y\cos(2y)))^2 dx^2}{(\cos(2y)(1 - 8y^2 + \cos(4x)) + 8y(\sin(2x) + \sin(2y)))^2}$$

$$+4\frac{(8y\cos(2y) - 4\sin(2x) + \sin(2y)(8y^2 + \cos(4x) - 3))^2 dy^2}{(\cos(2y)(1 - 8y^2 + \cos(4x)) + 8y(\sin(2x) + \sin(2y)))^2}.$$

Example 5

Consider $\sin(\theta(x,y)) = \tanh(C(x) + D(y))$ a family of solutions of the elliptic sine-Gordon. Then $C'(x) = c(x) = \sqrt{2}\tanh(\sqrt{2}x)$ and $D'(y) = d(y) = -\sqrt{2}\tanh(\sqrt{2}y)$. Then we obtain a solution θ of the elliptic sine-Gordon equation

$$\tan(\frac{\theta(x,y)}{2}) = \frac{\cosh(\sqrt{2}x) - \cosh(\sqrt{2}y)}{\cosh(\sqrt{2}x) + \cosh(\sqrt{2}y)}.$$

The corresponding solution w of the elliptic sinh-Gordon equation via the Bäcklund transformation is given by

$$\tanh(\frac{w(x,y)}{2}) = \frac{\sqrt{2}\sinh(\sqrt{2}y)}{\sqrt{2}\sinh(\sqrt{2}x) - 2\cosh(\sqrt{2}x)}.$$

Example 5

We can calculate the integrals I_1, I_2, I_3 and I_4

$$I_1(x) = \int_0^x \frac{\cosh^2(\sqrt{2}t) - 1}{\cosh^2(\sqrt{2}t) + 1} dt = x - \operatorname{arctanh}(\frac{\tanh(\sqrt{2}x)}{\sqrt{2}}),$$

$$I_2(x,y) = \int_0^y \frac{4\sqrt{2}f(x)g(x)\sinh(\sqrt{2}s)\cosh(\sqrt{2}s)}{(f^2(x) - 2\sinh^2(\sqrt{2}s))(g^2(x) + \cosh^2(\sqrt{2}s))} ds$$

$$= \frac{1}{2}\log\left(\frac{f^2(x) + 1 - \cosh(2\sqrt{2}y)}{2g^2(x) + \cosh(2\sqrt{2}y) + 1}\frac{2g^2(x) + 2}{f^2(x)}\right),$$

Construction of harmonic maps

Example 5

$$I_3(x) = \int_0^x e^{2x} e^{-2\,\mathrm{arctanh}\left(\frac{\tanh(\sqrt{2}x)}{\sqrt{2}}\right)} \frac{4\cosh(\sqrt{2}t)}{3+\cosh(2\sqrt{2}t)}\,dt$$

$$= \frac{4e^{2x}}{4\cosh(\sqrt{2}x) + 2\sqrt{2}\sinh(\sqrt{2}x)} - 1$$

$$I_4(x,y) = \frac{2g^2(x)+2}{f^2(x)} \int_0^y \frac{\sinh(\sqrt{2}s)(g^2(x) - \cosh^2(\sqrt{2}s))}{(g^2(x)+\cosh^2(\sqrt{2}s))^2}\,ds$$

$$= 2e^{2x}\Big(\frac{\cosh(\sqrt{2}y)(\sqrt{2}\sinh(\sqrt{2}x) - 2\cosh(\sqrt{2}x))}{2+\cosh(2\sqrt{2}x)+\cosh(2\sqrt{2}y)}$$

$$-\frac{\sqrt{2}\sinh(\sqrt{2}x) - 2\cosh(\sqrt{2}x)}{3+\cosh(2\sqrt{2}x)}\Big)$$

where $f(x) = \sqrt{2}\sinh(\sqrt{2}x) - 2\cosh(\sqrt{2}x)$, $g(x) = \cosh(\sqrt{2}x)$.

Example 5

To construct a harmonic map we use the initial conditions $S(0,0) = 1/2$ and $R(0,0) = 0$

$$S(x,y) = e^{2x}\frac{2 + 3\cosh(2\sqrt{2}x) - \cosh(2\sqrt{2}y) - 2\sqrt{2}\sinh(2\sqrt{2}x)}{2 + \cosh(2\sqrt{2}x) + \cosh(2\sqrt{2}y)}.$$

$$R(x,y) = 4e^{2x}\left(\frac{\cosh(\sqrt{2}y)(2\cosh(\sqrt{2}x) - \sqrt{2}\sinh(\sqrt{2}x))}{2 + \cosh(2\sqrt{2}x) + \cosh(2\sqrt{2}y)}\right) - 2$$

Then $u = R + iS$ a harmonic map from a surface to the upper-half plane equipped with Poincaré metric.

Future works

Non linear PDEs related to sine-Gordon

1. Double sine-Gordon and mixed sinh-cosh-Gordon.
2. Generalised sinh-Gordon with variable coefficient.
3. Boundary value problems.
4. Solutions using symmetries.

Harmonic maps

1. Harmonic maps in higher dimensions.
2. Harmonic maps between confromal manifolds.
3. Construction of biharmonic, exponential harmonic, p-harmonic mappings between Riemannian surfaces.

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Thank you for your attention!