Cartan's homotopy formula and the Linking number in rank-1 symmetric spaces

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joint with Evangelia Samiou

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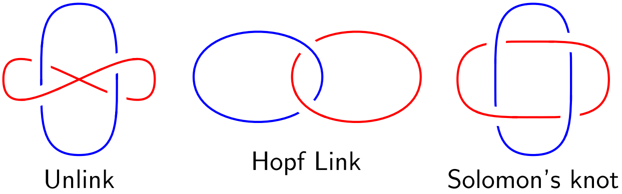
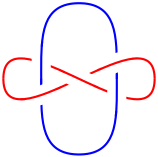
via crossings

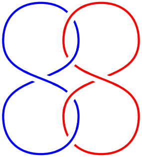


via crossings

Even if the linking number is 0, the link can not necessarily be unlinked:



Whitehead Link



as shown in a letter from Maxwell to Tait from 1867

$\mathrm{lk}(K, L) = 0$ but the unknotting number is 1.

Gauss' Linking Integral in $\mathbb{R}^n$

$K^k, L^l \subset \mathbb{R}^n$ closed oriented submanifolds,
$k + l + 1 = n$

Gauss-map:

$$\Phi\colon K \times L \to S^{n-1}$$

$$\Phi(k, l) = \frac{k - l}{\|k - l\|}$$

$$\mathrm{lk}(K, L) = \mathrm{degree}\,\Phi$$

$$\mathrm{lk}(K, L) = \frac{1}{\mathrm{vol} S^{n-1}} \int_{K \times L} \frac{\det(k - l, dk, dl)}{\|k - l\|^n}$$

Maxwell's equations

Magnetostatic Maxwell equations:

$$\mathrm{d}\mu = \omega \quad \Leftrightarrow \quad \operatorname{curl}\mu = \omega \quad \text{“Ampere's Law”}$$
$$\mathrm{d}^*\mu = 0 \quad \Leftrightarrow \quad \operatorname{div}\mu = 0 \quad \text{“no monopoles”}$$

for the stationary magentic field μ caused by a current density ω.

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for the stationary magentic field μ caused by a current density ω.

Biot-Savart law:

$$B(x) = \frac{1}{\mathrm{vol}S^2} \int_{\mathbb{R}^3} \frac{j(y) \times (x-y)}{\|x-y\|^3}\, d^3y\ .$$

If the current is confined to a loop $L^1 \subset \mathbb{R}^3$ and a magnetic unit monopole is moved along another loop $K^1 \subset \mathbb{R}^3$, then the work picked up is the linking number.

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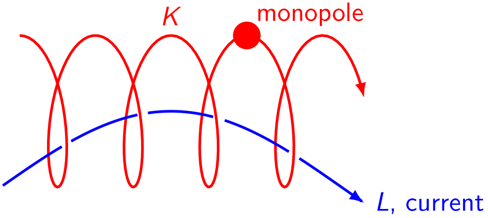

other methods to obtain Linking integrals

Cantarella, deTurck, Gluck: for 3-dimensional space forms, solve the magnetostatic Maxwell equations, Biot-Savart operator.

deTurck, Gluck: for linking in spheres, rewrite the linking number as the degree of a certain map

$$\underbrace{K * L}_{\text{join}} \to S^n$$

Shonkwiler, Vela-Vick: linking in "visible submanifolds" of Euclidean space, i.e. hypersurfaces whose radial projection to the unit sphere is a diffeomorphism.

via Thom Class, Poincaré Dual

X an oriented n-dimensional manifold.

$$K^k, L^\ell \subset X$$

disjoint, closed, connected, oriented submanifolds,

$$k+\ell+1=n \quad , \quad \dim\, K = k, \dim\, L = \ell$$

long exact cohomology sequence of the pair $X, X \setminus K$:

$$\begin{array}{ccccccc} & & \cong \mathbb{Z}u & & & & \\ H^{n-k}(X) & \xleftarrow[j^*]{} & H^{n-k}(X, X\setminus K) & \xleftarrow[\delta]{} & H^\ell(X\setminus K) & \xleftarrow[i^*]{} & H^\ell(X) \\ & \nwarrow & \uparrow{\scriptstyle \cup u} & & \downarrow{\scriptstyle i_L^*} & \swarrow & \\ & & H^0(K) & & H^\ell(L) & & \end{array}$$

via Thom Class, Poincaré Dual

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 & \nwarrow{\scriptstyle \text{should be } 0} & \uparrow{\scriptstyle \cup u} & & \downarrow{\scriptstyle i_L^*} & \swarrow{\scriptstyle \text{should be } 0} & \\
 & & H^0(K) & & H^{\ell}(L) & &
\end{array}
$$

u is the Thom class, or any Poincaré dual of K in X.

$$\mathrm{lk}(K, L) = \left\langle \delta^{-1} u \,\middle\|\, [L] \right\rangle$$

This is well-defined if the fundamental classes of K and L vanish in X.

For spheres K, L in rank-1 symmetric spaces X, or (compact) hyperbolic manifolds, this is "almost" always true,

Exceptions: $\mathrm{S}^2 \hookrightarrow \mathbb{C}P^n$, $\mathrm{S}^4 \hookrightarrow \mathbb{H}P^n$, $\mathrm{S}^8 \hookrightarrow \mathbb{O}P^2$

deRham Cohomology

For u we can take any Poincaré dual of K in X. This is any form $u \in \Omega^{n-k}(X)$ so that

1. $du = 0$
2. $\displaystyle\int_K \alpha = \int_X u \wedge \alpha$ for all $\alpha \in \Omega^k(X)$ with $d\alpha = 0$

If the support of u is in a tubular neighbourhood of K, then such a form represents the Thom-class of the normal bundle of K in X corresponding to the orientations. This exists for any tubular neighbourhood.

We can localize u, i.e. choose it with support in an arbitrarily small tubular neighbourhood of K.

A formula for the linking number now amounts to finding a right inverse to the Cartan differential

$$\mathrm{d}\colon \Omega^\ell(X) \to \Omega^{\ell+1}(X)\ .$$

$$\mathrm{lk}(K,L) = \int_L \mathrm{d}^{-1} u$$

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$$(\mathrm{d}^{-1}\omega)_x = \int_M \lambda(d(x,y)) \mathcal{L}^*_{yx} \iota_{T_{yx}} \mathcal{P}^*_{yx} \omega_y \ dy \ , \qquad (1)$$

for symmetric spaces of rank 1.

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Theorem

*Let M be a connected oriented Riemannian manifold and $\mathcal{L} \in Hom(TM, TM)$ and $\lambda \in C^\infty(\mathbb{R}^+)$ are as in the general form for d^{-1} in (**??**). Let $K, L \subset M$ be disjoint, connected closed oriented submanifolds of dimensions k, l respectively, with $k + l + 1 = \dim M$. If the fundamental classes of K and L vanish in M, then the linking number of K with L is*

$$\mathrm{lk}(K,L) = \int_K \int_L \lambda(d(x,y))\mathrm{vol}^M_y(\mathrm{d}L_y \wedge \mathcal{P}_{yx}(T_{yx} \wedge \Lambda^k \mathcal{L}_{yx}\mathrm{d}K_x)) \ d^l y \ d^k x \ .$$

Cartan's Formula

In the sequel we will derive formulas for $\mathrm{d}^{-1}u$ is by integrating Cartan's "magic" Formula

$$L_T\omega = \iota_T \mathrm{d}\omega + \mathrm{d}(\iota_T\omega) \ .$$

1. T a vector field
2. L_T the Lie derivative in direction of this vector field .
3. $(\iota_T\omega)(v_1, \ldots, v_r) = \omega(T, v_1, \ldots, v_r)$, "contraction"

If $T(p) = \frac{d}{dt}\phi(t)(p)$ for a flow ϕ, then

$$L_T\omega = \frac{d}{dt}\phi(t)^*\omega$$

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If $\mathrm{d}\omega = 0$ then integrating this gives

$$\phi(t)^*\omega = \mathrm{d}\int_s^t \iota_T \phi(\tau)^*\omega \; d\tau + \phi(s)^*\omega$$

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Assuming that t is some suitable end-time of the flow ϕ, i.e.

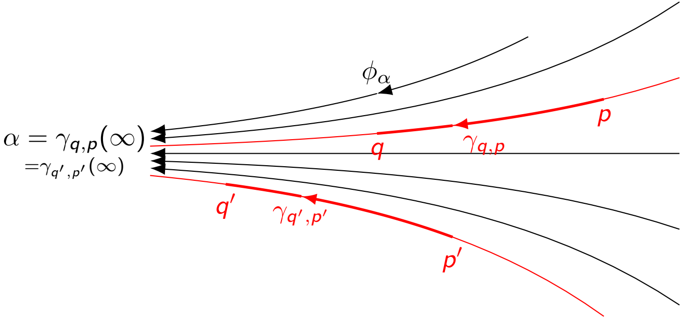
$$\phi(t)^*\omega = 0$$

and $s = 0$, we get

$$\omega = \mathrm{d}\mu_\phi \quad \text{with} \quad \mu_\phi = -\int_0^t \iota_T\phi(\tau)^*\omega \, d\tau$$

Negatively curved symmetric spaces

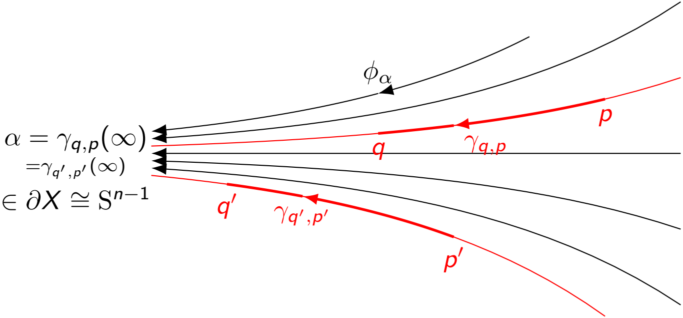
In negatively curved symmetric spaces we can explicitly compute this for the negative gradient flow of Busemann functions. For each ideal point α we have a flow ϕ_α,



In these spaces for any two points p, q there is a unique geodesic $\gamma_{q,p}$ from p to q,

$$\gamma_{q,p}(0) = p \quad , \quad \gamma_{q,p}(d(p,q)) = q \quad , \quad T_{q,p} := \gamma'_{q,p}(0) \ .$$

Negatively curved symmetric spaces



We now average over all these flows, i.e. over the ideal boundary $\partial X \cong \mathrm{S}^{n-1}$:

$$\mu := \frac{1}{\mathrm{vol}\mathrm{S}^{n-1}} \int_{\partial X} \mu_{\phi_\alpha} \, d\alpha \; .$$

Theorem

Let M be a negatively curved symmetric space and let $\omega \in \Omega^{k+1}$ be closed, i.e. $\mathrm{d}\omega = 0$, and of compact support. Then we have $\omega = \mathrm{d}\mu$ with $\mu \in \Omega^k(M)$ defined by

$$\mu_x = \frac{-1}{\mathrm{vol} \mathrm{S}^{n-1}} \int\limits_M \frac{2^m \; \mathcal{L}^*_{yx} \iota_{T_{yx}} \mathcal{P}^*_{yx} \omega}{\sinh(2\mathrm{d}(x,y))^m \sinh(\mathrm{d}(x,y))^{\dim \; M-m-1}} \; dy \;,$$

and

$$\mathcal{L}_{yx} = e^{-d(y,x)} \sqrt{J_{yx}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-2d(y,x)} \mathrm{id}_m & 0 \\ 0 & 0 & e^{-d(y,x)} \mathrm{id}_{\dim \; M-m-1} \end{pmatrix}$$

$\mathbb{R}^n$ and H^n

Euclidean space $\mathbb{R}^n$: Riesz-potential

$$\mu_p(v_1, \ldots, v_k) = \frac{-1}{\mathrm{vol} S^{n-1}} \int_{\mathbb{R}^n} \frac{\omega_q(p - q, v_1, \ldots, v_k)}{\|p - q\|^n} d^n q$$

hyperbolic space H^n:

$$\mu_p(v_1, \ldots, v_k) = \frac{-1}{\mathrm{vol} S^{n-1}} \int_{\mathbb{R}^n} \frac{e^{-kd(p,q)} \omega_q(T_{p,q}, \mathcal{P}_{q,p} v_1, \ldots, \mathcal{P}_{q,p} v_k)}{\sinh(d(p,q))^{n-1}} d^n q$$

where $\mathcal{P}_{q,p}$ denotes the parallel transport from p to q along the geodesic $\gamma_{q,p}$.

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where $\mathcal{P}_{q,p}$ denotes the parallel transport from p to q along the geodesic $\gamma_{q,p}$.

The linking number in H^n becomes

$$\mathrm{lk}(K, L) = \frac{-1}{\mathrm{vol} \mathrm{S}^{n-1}} \int_K \int_L \frac{e^{-kd(y,x)} \mathrm{vol}_y^M (\mathrm{d}L_y \wedge \mathcal{P}_{yx}(T_{yx} \wedge \mathrm{d}K_x))}{\sinh(\mathrm{d}(x,y))^{\dim M - m - 1}} d^l y d^k x \, .$$

$\mathbb{C}H^n$

Complex hyperbolic space $\mathbb{C}H^n$:

$$\mu_p(v_1,\ldots,v_k) = \frac{-1}{\mathrm{vol} S^{n-1}} \int_{\mathbb{C}H^n} \frac{2\,\omega_q(T_{p,q}, Q_{p,q}v_1,\ldots,Q_{p,q}v_k)}{\sinh(2d(p,q))\sinh(d(p,q))^{n-2}} d^n q$$

where

$$Q_{p,q}v = \begin{cases} e^{-d(p,q)}P_{q,p}v & \text{if } v \perp iT \\ e^{-2d(p,q)}P_{q,p}v & \text{if } v = iT \end{cases}$$

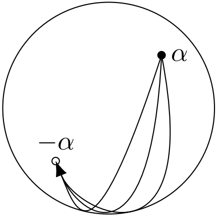
$$Q_{p,q} = \begin{pmatrix} e^{-2d(p,q)} & 0 & \cdots & 0 \\ 0 & e^{-d(p,q)} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e^{-d(p,q)} \end{pmatrix} \circ P_{q,p}$$

A similar formula holds for quaternionic hyperbolic space $\mathbb{H}H^n$ and the Cayley hyperbolic plane.

Spheres

For $\alpha \in \mathrm{S}^n$, replace the Busemann function by the distance from $-\alpha$. Work with the local the flow ϕ_α, so that

$$\frac{d}{dt}\phi_\alpha(t) = -\operatorname{grad} \underbrace{d_{-\alpha}}_{\text{distance from}-\alpha}$$



Now we need to integrate over $\alpha \in \mathrm{S}^n$ to remove the singularity.

Spheres

Theorem

Let $\omega \in \Omega^{k+1}(\mathrm{S}^n)$ be exact (equivalently closed and $k < n-1$ or $k = n-1$ and $\int_{\mathrm{S}^n} \omega = 0$). Let $\mu \in \Omega^k(\mathrm{S}^n)$ be defined by

$$\mu_x = \int_{\mathrm{S}^n} \lambda(d(x,y))\ \iota_{T_{yx}} \mathcal{P}^*_{yx}\omega\ d^n y$$

where

$$\lambda(d) = \frac{-1}{\sin(d)^{n-1}\mathrm{vol}\mathrm{S}^n} \int_d^\pi \sin(s)^{n-1-k} \sin(s-d)^k\ ds\ .$$

Then

$$\mathrm{d}\mu = \omega$$

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Then

$$\mathrm{d}\mu = \omega$$

Corollary

The linking number of closed oriented submanifolds $K^k, L^l \subset \mathrm{S}^n$, $k+l+1 = n$ is

$$\mathrm{lk}(K,L) = \int_K \int_L \frac{\lambda(d(x,y))}{\sin(d(x,y))} \det(\mathrm{d}L_y, -x, \mathrm{d}K_x, y)\ dxdy$$

Spheres

this coincides with the mapping degree obtained by deTurck and Gluck.

Projective Spaces $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$, $\mathbb{O}P^2$

In place of the Busemann functions we use the distance function from a projective r-plane V,

$$\mathbb{K}P^r \cong V \subset \mathbb{K}P^n$$

to its focal locus

$$\overline{V} \cong \mathbb{K}P^{n-r-1} .$$

Then we average over the Grassmannian of projective r-planes in $\mathbb{K}P^n$.

Projective Spaces $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$, $\mathbb{O}P^2$

Theorem

Let $\omega \in \mathrm{d}\Omega^k(\mathbb{K}P^n) \subset \Omega^{k+1}(\mathbb{K}P^n)$. Then $\omega = \mathrm{d}\mu$ with $\mu \in \Omega^k(\mathbb{K}P^n)$ given by

$$\mu_x = \frac{-1}{\mathrm{vol}\, G_r(\mathbb{K}P^n)} \int\limits_{\mathbb{K}P^n} \int\limits_{d(x,y)}^{\frac{\pi}{2}} \frac{1}{\mathrm{Jac}_x(y,s)} \int\limits_{\substack{V \ni \gamma_{yx}(s) \\ V \perp_{\mathbb{K}} \gamma'_{xy}(s)}} \Lambda^k \mathcal{L}_{yx}^{V\,*}\, dV\, ds\, \iota_{T_{yx}} \mathcal{P}_{yx}^{*} \omega\, dy$$

Example: $\mathbb{C}P^2$, d^{-1}

Let $\omega \in \mathrm{d}\Omega^1(\mathbb{C}P^2) \subset \Omega^2(\mathbb{C}P^2)$. Then $\omega = \mathrm{d}\mu$ for $\mu \in \Omega^1(\mathbb{C}P^2)$ given at $x \in \mathbb{C}P^2$ by

$$\mu_x = \frac{-2}{\pi^2} \int_{\mathbb{C}P^2} \frac{\iota T_{yx}}{\sin(2d(x,y))\sin^2(d(x,y))} \mathcal{L}^*_{yx} \mathcal{P}^*_{yx} \omega \; dy$$

with

$$\mathcal{L}_{yx} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & L^0_{yx} & 0 \\ 0 & 0 & L^1_{yx}\ \mathrm{id}_2 \end{pmatrix}$$

$$L^0_{yx} = \frac{(\pi - 2d(x,y))\ \sin(2d(x,y)) + 2\cos(2d(x,y)) + 2}{8}$$

$$L^1_{yx} = \frac{\cos(3d(x,y)) + (4d(x,y) - 2\pi)\sin(d(x,y)) + 7\cos(d(x,y))}{16}$$

Example: $\mathbb{C}P^2$, linking number

Let $K^1, L^2 \subset \mathbb{C}P^2$ be connected closed oriented nullhomologous submanifolds of dimension 1 and 2 respectively.

$$\mathrm{lk}(K,L) = \frac{-2}{\pi^2} \int_K \int_L \frac{\mathrm{vol}_y^{\mathbb{C}P^2}(\mathrm{d}L_y \wedge \mathcal{P}_{y,x}(T_{y,x} \wedge \mathcal{L}_{y,x}\mathrm{d}K_x))}{\sin(2d(x,y))\sin^2(d(x,y))}\, d^l y\, d^k x\ .$$

Splitting $\mathrm{d}K_x = \kappa_0(x) iT + \kappa_1$ and $\mathrm{d}L_y = \lambda_0 \wedge iT + \lambda_1$ into components perpendicular respectively parallel to iT_{yx}, this becomes

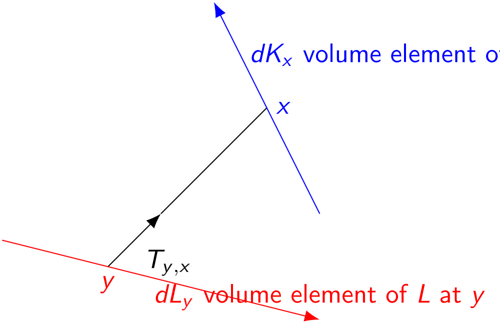
$$\mathrm{lk}(K,L) = \frac{-2}{\pi^2} \int_K \int_L L^0_{y,x} \mathrm{vol}_y^{\mathbb{C}P^2}(\lambda_1(y) \wedge \mathcal{P}_{y,x}(T_{y,x} \wedge \kappa_0(x) iT))$$

$$+ L^1_{y,x} \mathrm{vol}_y^{\mathbb{C}P^2}(\lambda_0(y) \wedge iT \wedge \mathcal{P}_{y,x}(T_{y,x} \wedge \kappa_1(x)))\, d^l y\, d^k x\ .$$

in all cases

For almost all pairs $(x, y) \in X \times X$ we have a unique minimizing geodesic $\gamma_{y,x}$. There is a universal function W so that

$$\mathrm{lk}(K, L) = \int_{K \times L} W(d(x, y), dL_y, T_{x,y}, \mathcal{P}_{y,x} dK_x) \ .$$



Cross-helicity

$$c(\omega, \alpha) = \int_M \mathrm{d}^{-1}\omega \wedge \alpha$$

where $\mu = \mathrm{d}^{-1}\omega \in \Omega^{k-1}(M)$ is any form with $\mathrm{d}\mu = \omega$.

Cross-helicity

$$c(\omega,\alpha) = \int_M \mathrm{d}^{-1}\omega \wedge \alpha$$

can be estimated by Youngs convolution inequality:

$$|c(\omega,\alpha)| \leq \|\omega\|_p \, \|\alpha\|_{p'} \, \left\|\text{kernel of } \mathrm{d}^{-1}\right\|_q \;,$$

if

$$2 = \frac{1}{p} + \frac{1}{p'} + \frac{1}{q} \quad \text{and} \quad q < \frac{n}{n-1} \;.$$

Hopf invariant

$f\colon \mathrm{S}^{2k-1} \to \mathrm{S}^k$ and let $\omega \in \Omega^k(\mathrm{S}^k)$ be any volume form, i.e. $\int_{\mathrm{S}^k} \omega = 1$. The Hopf invariant $h(f)$ of this map is the helicity of $f^*\omega$. i.e. its cross-helicity with itself,

$$h(f) = c(f^*\omega, f^*\omega) = \int_{\mathrm{S}^{2k-1}} \mathrm{d}^{-1} f^*\omega \wedge f^*\omega \ .$$

For $p = 2 = p'$ and $q = 1$ the conditions above are satisfied

$$|h(f)| \leq \|f^*\omega\|_2^2 \left\|\text{kernel of } \mathrm{d}^{-1}\right\|_1 \ .$$

Hopf invariant

We now assume that S^{2k-1} carries its standard symmetric metric. On S^k consider an arbitrary Riemannian metric with volume 1, and volume form ω. Then

$$|f^*\omega|^2 = (Jf)^2 = \det(df df^*)$$

is the Jacobian where the adjoint df^* is computed with respect to the Riemannian metrics. Since

$$\det(df df^*) \le \frac{1}{k^k}\,\mathrm{trace}(df df^*)^k = \frac{1}{k^k}\,\mathrm{trace}(df^* df)^k$$

the $2k$-energy $E_{2k}(f) = \|df\|_{2k}^{2k}$ dominates the 2-norm of the Jacobian,

Hopf invariant

$$\|Jf\|_2^2 = \int_{S^{2k-1}} \det(df df^*) \le \frac{1}{k^k} \int_{S^{2k-1}} \mathrm{trace}(df df^*)^k = \frac{\|df\|_{2k}^{2k}}{k^k} \ .$$

Computing $\left\|\text{kernel of } \mathrm{d}^{-1}\right\|_1$ gives the following.

Energy estimates for the Hopf invariant

Theorem

Let $f : \mathrm{S}^{2k-1} \to \mathrm{S}^k$ be a smooth map. Assume S^{2k-1} is equipped with the standard metric, and S^k carries any Riemannian metric of volume 1. Then the Hopf invariant of f is estimated by the $2k$-energy of f,

$$|h(f)| \leq \|df\|_{2k}^{2k} \times \frac{1}{k^k} \frac{\mathrm{vol}(\mathrm{S}^{2k-2})}{\mathrm{vol}(\mathrm{S}^{2k-1})} \int_0^{\pi} \int_r^{\pi} \sin(s)^{k-2} \sin(s-r)^k \, ds \, dr \ .$$

Examples:

$$f_2 : \mathrm{S}^3 \to \mathrm{S}^2 \text{ has } |h(f_2)| \leq \frac{\pi}{8} E_4(f_2)$$
$$f_4 : \mathrm{S}^7 \to \mathrm{S}^4 \text{ has } |h(f_4)| \leq \frac{3\pi}{2560} E_8(f_4) \ .$$