Moduli spaces of flat connections and the Atiyah-Bott classes

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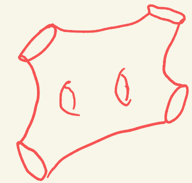


Moduli spaces of flat connections on surfaces

G – compact Lie group with $\pi_0 G = \pi_1 G = \{1\}$ (e.g. SU_N)

$\mathfrak{g}$ – $\mathrm{Lie}(G)$, with positive definite integral $\ell \in \mathrm{Sym}^2(\mathfrak{g}^*)^{\mathfrak{g}}$ "level"

V – faithful representation of G (e.g. $\mathbb{C}^N$)

Σ – compact connected oriented surface with $\partial\Sigma \cong S^1$ parametrized

A <u>connection</u> (on $\Sigma \times V$) is an operator

$$d_A = d + A : \Omega^\bullet(\Sigma, V) \longrightarrow \Omega^{\bullet+1}(\Sigma, V)$$

where $A \in \Omega^1(\Sigma, \mathfrak{g})$

$$\boxed{\mathcal{A}(\Sigma) = \{ d_A \mid A \in \Omega^1(\Sigma, \mathfrak{g}) \}}$$

curvature $\quad F_A = d_A^2 = dA + \frac{1}{2}[A, A] \in \Omega^2(\Sigma, \mathfrak{g})$

$\mathrm{Map}(\Sigma, G) \circlearrowright \mathcal{A}(\Sigma)$ by <u>gauge transformations</u>

$$g : d_A \longmapsto g \circ d_A \circ g^{-1} = d_{A^g}, \qquad A^g = gAg^{-1} - dg\, g^{-1}$$

$$F_{A^g} = g F_A g^{-1}$$

d_A is <u>flat</u> if its <u>curvature</u> $F_A = 0$

$$\boxed{\mathcal{A}_{flat}(\Sigma) = \{ d_A \mid A \in \Omega^1(\Sigma, \mathfrak{g}),\ d_A^2 = 0 \}}$$

Action of $Map_\partial(\Sigma, G) = \{ g \in Map(\Sigma, G) \mid g|_{\partial\Sigma} = 1 \}$ free and proper.

The <u>moduli space of flat connections with framing along $\partial\Sigma$</u> is

$$\mathcal{M}_\Sigma = \mathcal{A}_{flat}(\Sigma) / Map_\partial(\Sigma, G)$$

The moduli space of flat connections with framing along $\partial\Sigma$ is

$$\mathcal{M}_\Sigma = \mathcal{A}_{flat}(\Sigma) \big/ \mathrm{Map}_\partial(\Sigma, G) \qquad (*)$$

- smooth infinite-dim. manifold
- weakly symplectic ($(*)$ is a symplectic quotient, $\omega_{AB}(a,b) = \int_\Sigma \ell(a,b)$, [Atiyah-Bott] [Donaldson])
- Hamiltonian action of $\mathrm{Map}(\Sigma, G) \big/ \mathrm{Map}_\partial(\Sigma, G) \cong LG = \mathrm{Map}(S^1, G)$

proper moment map

$$\mu: \mathcal{M}_\Sigma \longrightarrow \Omega^1(S^1, \mathfrak{g}) \subseteq L\mathfrak{g}^*$$

$$[d_A] \longmapsto A\big|_{\partial\Sigma}$$

Hamiltonian loop group spaces

A <u>proper Hamiltonian LG-space</u> is an infinite-dim. weakly symplectic manifold (M,ω) carrying a Hamiltonian action of $LG = \mathrm{Map}(S^1, G)$ with proper moment map $\mu : M \longrightarrow \Omega^1(S^1, \mathfrak{g}) \subseteq L\mathfrak{g}^*$ [Meinrenken-Woodward]

$$\iota(X)\omega = -d\int_{S^1} \ell(\mu, X) \qquad \forall X \in L\mathfrak{g} = \Omega^0(S^1, \mathfrak{g})$$

<u>Example</u> orbits of $LG \circlearrowright \Omega^1(S^1, \mathfrak{g})$ (coadjoint orbits)

<u>Rmk</u> [M-W] use Sobolev spaces of loops, and M is a Banach manifold.

Suppress this for simplicity.

Motivation

$$\mathcal{M}_\Sigma = \mathcal{A}_{flat}(\Sigma) / \mathrm{Map}_\partial(\Sigma, G) \xrightarrow{\mu} \Omega^1(S^1; \mathfrak{g})$$

→ symplectic quotient $\mathcal{M}_{\Sigma,0} = \mu^{-1}(0)/G$ is the moduli space of flat connections on $\Sigma \cup_{\partial\Sigma} \mathbb{D}$, known to algebraic geometers as the moduli space of semistable $G_{\mathbb{C}}$-bundles

→ topology of $\mathcal{M}_{\Sigma,0}$ can be studied via G-equivariant topology of $\mathcal{M}_\Sigma$

Example Verlinde formula for the index of line bundles on $\mathcal{M}_{\Sigma,0}$
(Jeffrey-Kirwan, Bismut-Labourie, Alekseev-Meinrenken-Woodward, ...)

→ in algebraic geometry, Teleman-Woodward proved a generalization of the Verlinde formula covering a broad collection of K-theory classes

K - theory

H - Hilbert space, $\dim(H) = \infty$

$$\mathcal{F} = \{ T \in \mathcal{B}(H) \mid T \text{ is Fredholm} \} \qquad \text{(norm topology)}$$

$$K(X) = [X, \mathcal{F}] \qquad \text{(topologist's / non-compact support)}$$

<u>Rmk</u> If X is loc. compact, $RKK(X; C_0(X), C_0(X))$ in Kasparov's notation.

<u>Rmk</u> Variants with $T : H_1 \longrightarrow H_2$.

<u>Rmk</u> G compact Lie group, $K_G(X)$.

$\bar{\partial}$ operators

Fix a complex structure on Σ.

$$d_A \rightsquigarrow (d_A)^{0,1} = \bar{\partial}_A : \Omega^0(\Sigma, V) \longrightarrow \Omega^{0,1}(\Sigma, V)$$

family $\{ \bar{\partial}_A \mid d_A \in \mathcal{A}_{flat}(\Sigma) \}$ is gauge-equivariant, but does not define a K-class because

$$\partial\Sigma \neq \phi \implies \bar{\partial}_A \text{ not Fredholm}$$

Boundary conditions

On the complex disk $\mathbb{D}$, $\ker(\bar{\partial})$ is infinite dimensional

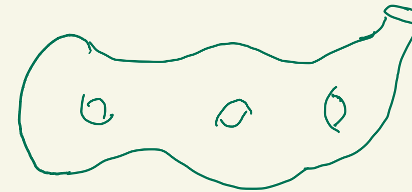
$$1, z, z^2, z^3, \ldots$$

To cut down to finite dimensions, impose a <u>boundary condition</u>

$$f\big|_{\partial\mathbb{D}} \in B_{<0}\left(\tfrac{1}{i}\partial_\theta\right) = \left\{ \sum_{n<0} a_n e^{in\theta} \right\}$$

then $\ker\left(\bar{\partial}^b : \Omega^0(\mathbb{D}; B_{<0}(\tfrac{1}{i}\partial_\theta)) \longrightarrow \Omega^{0,1}(\mathbb{D})\right) = 0$

On a general Σ, we do the same: the operator



Σ

$$\bar{\partial}_A^b : \Omega^0\left(\Sigma, V; B_{<0}\left(\tfrac{1}{i}\partial_\theta\right)\right) \longrightarrow \Omega^{0,1}(\Sigma, V) \qquad (*)$$

is Fredholm.

Rmk: This is essentially the Atiyah-Patodi-Singer boundary condition. Note however that we take $B_{<0}\left(\frac{1}{i}\partial_\theta\right)$ to be independent of A.

(*) Slightly cleaner if we use a spin str on Σ. We assume this below.

The family $\{ \bar{\partial}_A^b \mid d_A \in \mathcal{A}(\Sigma) \}$ is not gauge-equivariant:

$$B_{<0}\left(\tfrac{1}{i}\partial_\theta\right) \xrightarrow{g \in \mathrm{Map}(\Sigma, G)} B_{<0}\left(g_\partial \circ \tfrac{1}{i}\partial_\theta \circ g_\partial^{-1}\right), \qquad g_\partial = g|_{\partial\Sigma}$$

Preserved by $g : \Sigma \to G$ such that $g|_{\partial\Sigma}$ is constant, and $\dfrac{\mathrm{Map}_{const}(\Sigma, G)}{\mathrm{Map}_\partial(\Sigma, G)} \cong G$.

Thm [L] The family $\{ \bar{\partial}_A^b \}$ descends to a continuous family of Fredholm operators parametrized by $\mathcal{M}_\Sigma = \dfrac{\mathcal{A}_{flat}(\Sigma)}{\mathrm{Map}_\partial(\Sigma, G)}$.

Notation: $E^\Sigma V \in K_G(\mathcal{M}_\Sigma)$

The index homomorphism

$\mathcal{M}$ – proper Hamiltonian LG-space, such as $\mathcal{M}_\Sigma$

Next, define a homomorphism

T – max. torus of G

$\Lambda = \mathrm{Hom}(T, U(1))$

$$\mathrm{index}_\ell : K_T(\mathcal{M})_{gc} \longrightarrow R^{-\infty}(T) = \mathbb{Z}^\Lambda$$

gc is a "growth condition" – to be explained

Thm [L] For $\ell \gg 0$, the coefficient of e^λ in $\mathrm{index}_\ell(\Psi)$ computes a K-theoretic intersection pairing on a finite dimensional symplectic quotient of $\mathcal{M}$.

A triple (X, ξ, f) where

X – finite even dim. T-equiv. Spin^c

$\xi \in K_{T,c}(X)$ (compactly supported K-theory)

$f : X \longrightarrow M$ T-equiv.

pairs with $K_T(M)$: $\langle [(X,\xi,f)], \psi \rangle = \mathrm{index}_T^X(\xi \otimes f^*\psi) \in R(T)$

Definition of index_ℓ involves a non-compactly supported ξ.

→ pairing only defined for subring $K_T(M)_{gc}$

→ output in $R^{-\infty}(T)$

Construction of (X, ξ, f)

<u>Thm</u> [L-Meinrenken-Song] $\exists$ $\dim(\mathfrak{g})$-dim. submanifold $\mathfrak{t} \subseteq \mathcal{U} \subseteq \Omega^1(S^1, \mathfrak{t})$ which is transverse to gauge orbits (canonical up to homotopy).

$$X = \mu^{-1}(\mathcal{U}) \quad , \quad f = \text{inclusion} \, , \quad \xi = \mu^* \operatorname{Thom}(\mathcal{U}, \mathfrak{t}^*)$$

$$X \circlearrowleft \operatorname{Norm}_{LG}(\mathfrak{t}^*) \underset{\substack{\text{finite} \\ \text{index}}}{\supseteq} N = \left\{ \begin{matrix} \text{closed} \\ \text{geodesics} \\ \text{in } T \end{matrix} \right\} \cong T \times \Pi \quad , \quad \Pi = \operatorname{Hom}(S^1, T)$$

<u>Thm</u> [L-Meinrenken-Song] $\exists$ canonical Spin^c structure on X induced by a compatible almost complex structure on the symplectic manifold $\mathcal{M}$.

Growth condition

$N = \left\{ \begin{smallmatrix} \text{closed} \\ \text{geodesics} \\ \text{in } T \end{smallmatrix} \right\}$

Assume the $K_T(M)$ - class represented by a smooth T-equiv. family of operators with compact resolvent:

$$\begin{tikzcd} \mathcal{E}^+ \supseteq \mathrm{dom}(Q) \arrow[rr, "Q"] \arrow[dr] & & \mathcal{E}^- \arrow[dl] \\ & M & \end{tikzcd}$$

$(\mathcal{E}, Q)$ <u>satisfies the gc</u> if

① $\mathcal{E}|_X$ is N - equiv., using the norm topology on $\mathrm{Aut}(\mathcal{E}|_X)$

② $\exists$ N - equiv. connection $\nabla^{\mathcal{E}|_X}$ preserving $\mathrm{dom}(Q)|_X$ such that

$\nabla^{\mathcal{E}|_X} Q|_X$ is a bounded section of $T^*X \otimes \mathcal{B}(\mathcal{E}|_X)$

Rmk $\mathrm{dom}(Q)|_X \subseteq \mathcal{E}^+|_X$ not required to be N-invariant.

Finite rank N-equiv. vector bundles satisfy the g.c.

Thm [L] The family of Fredholm operators on $\mathcal{M}_\Sigma$ induced by $\{ \bar{\partial}^b_A \mid d_A \in \mathcal{A}_{flat}(\Sigma) \}$ satisfies the g.c.

There are K-classes that do not satisfy the g.c.

Rmk The 'size' of $K_T(\mathcal{M})_{gc}$ in $K_T(\mathcal{M})$ remains unclear (to me).

Thm [L] There is a homomorphism

$$\mathrm{index}_\ell : K_T(\mathcal{M})_{gc} \longrightarrow R^{-\infty}(T) = \mathbb{Z}^\Lambda$$

that takes $[(\mathcal{E}, Q)]$ to the L^2-index of $\mathbb{D}^{\mathcal{L}}_{X,\xi} \hat{\otimes}_{\nabla^{\mathcal{E}}} 1 + 1 \hat{\otimes} Q_x$

$\mathcal{L}$ = prequantum line bundle on $\mathcal{M}$, at level ℓ

i.e. $c_1(\mathcal{L}) = [\omega] \in H^2(\mathcal{M})$, $\mathcal{L} \circlearrowleft \widehat{LG}^{(\ell)}$

Rmk $\mathcal{L}$ does not satisfy the gc.

Rmk Generalizes [L - Song], which treated finite rank N-equiv. vector bundles.

Example

$\Sigma = \mathbb{D}$, $\quad \mathcal{M}_{\mathbb{D}} \cong LG/G$ ("smooth affine Grassmanian")

$$X \sim \mathcal{M}_{\mathbb{D}}^{T \times S^1_{rot}} \cong \Pi = \mathrm{Hom}(S^1, T) \xhookrightarrow{\ \ell\ } \Lambda = \mathrm{Hom}(\Pi, \mathbb{Z})$$

Π-orbit of trivial connection on $\mathbb{D}$

$$\mathrm{index}_k(E^{\mathbb{D}} V) = J \sum_{\eta \in \Pi} e^{\eta} \cdot \mathrm{index}\left(\bar{\partial}^{b}_{\tilde{\eta}}\right), \quad J = \text{Weyl denominator for } G$$

$$= J \sum_{\eta \in \Pi} e^{\eta} \langle d\chi_V, \eta \rangle$$

where $\chi_V(u) = \mathrm{Tr}_V(\rho_V(u))$, $u \in T$ character of (V, ρ_V)

$$\operatorname{index}_k(E^{\mathbb{D}}V) = J \sum_{\eta \in \Pi} e^{\eta} \langle d\chi_V, \eta \rangle$$

Consider the generating series

$$e^{tE^{\mathbb{D}}V} = \sum_{m \geq 0} \frac{t^m}{m!} (E^{\mathbb{D}}V)^{\otimes m} \in K_T(M_{\mathbb{D}})[\![t]\!] \otimes \mathbb{Q}$$

$$\Rightarrow \operatorname{index}_k\left(e^{tE^{\mathbb{D}}V}\right)(u) = J(u) \sum_{\eta \in \Pi} \Phi_t(u)^{\eta} \quad , \quad \Phi_t(u) = u\, e^{-t\nabla \chi_V(u)}$$

(formal diffeomorphism of T)

$$\overset{\text{Poisson}}{=} \frac{J(u)}{|T_\ell|} \sum_{\varsigma \in T_\ell} \frac{\delta_{\varsigma_t}(u)}{\det\left(1 - t\nabla^2 \chi_V(u)\right)}$$

where $\Phi_t(\varsigma_t) = \varsigma$ and $T_\ell = \ell^{-1}(\Lambda)/\Pi \subseteq T$.

Remarks on $K_T(\mathcal{M}_{\mathbb{D}})$

$$i^* : K_T(\mathcal{M}_{\mathbb{D}}) \longrightarrow K_T(\mathcal{M}_{\mathbb{D}}^{T \times S^1_{rot}}) = K_T(\Pi) = \mathrm{Fun}(\Pi, R(T))$$

is injective, image consists of

$$\psi : \Pi \to R(T) \quad \text{st} \quad (1 - e^{\alpha})^m \text{ divides } \Delta^m_{\alpha^\vee} \psi$$

finite diff. operator

for each $m > 0$ and root α of G.

[Lam - Schilling - Shimozono], [Kostant - Kumar]

$$\left(K_T(\mathcal{M}_{\mathbb{D}}) \cong \prod_{\eta \in \Pi} R(T) \quad \text{as } R(T)\text{-mod., Schubert basis} \right)$$

Non-abelian localization

Recall index_k takes $\psi = [(\mathcal{E}, Q)] \in K_T(M)_{gc}$ to the L^2-index of

$$D = \not{D}^{\mathcal{L}}_{X,\xi} \hat{\otimes}_{\nabla^{\mathcal{E}}} 1 + 1 \hat{\otimes} Q_X$$

Let $D_s = D + s\, c(\kappa)$, $\kappa = \omega^{-1}(d\|\mu\|^2) / (1 + \|\mu\|^2)^{1/2}$

By considering $s \longrightarrow \infty$ get

cf.

[Witten], [Bismut-Lebeau], [Tian-Zhang], [Ma-Zhang], [Paradan], [Hochs-Song], [L-Song], ...

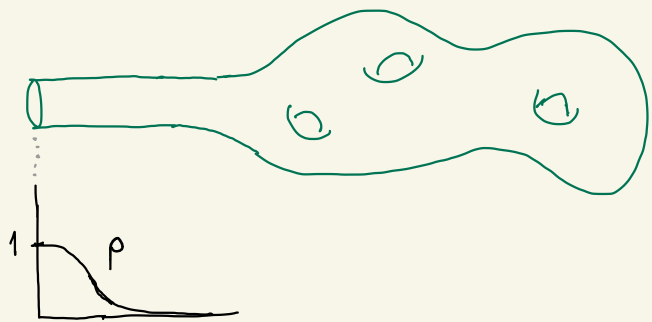
<u>Thm</u> [L] $\mathrm{index}_\ell(\psi) = \sum_{v \in \mathrm{Crit}(\|\mu\|^2)} \mathrm{index}_{\ell,v}(\psi|_{C_v})$

<u>Rmk</u> $C_v \cap \mathrm{supp}(\xi)$ compact, sum over v infinite.

Remarks on K

For $\mathcal{M}_\Sigma$

Σ



the flow of $\rho\,\partial_\theta \in \mathfrak{X}(\Sigma)$

induces a flow on $\mathcal{M}_\Sigma$

gives the flow of K

(up to reparametrization)

For $\Sigma = \mathbb{D}$, orbits close, S^1_{rot}-action. The sum over $\Pi = \mathcal{M}^{T \times S^1_{rot}}$

was the non-abelian localization formula in that case.

Main theorem Under further assumptions on $\Psi \in K_T(\mathcal{M})_{gc}$ one gets both a Kirillov-Berline-Vergne formula & Atiyah-Bott-Segal-Singer formula.

We'll just state the latter for the special case $\mathcal{M} = \mathcal{M}_\Sigma$, $\Psi = E^\Sigma V$.

Thm [L]

$$\operatorname{index}_\ell(E^\Sigma V) = \frac{1}{J} \sum_{\mathfrak{f} \in T_\ell^{reg}} \left(\frac{J^2}{|T_\ell| \det(1 - t\nabla^2 \chi_V)} \right)^{1-\text{genus}(\Sigma)} \delta_{\mathfrak{f}_t}$$

where $\Phi_t(\mathfrak{f}_t) = \mathfrak{f}$, $\Phi_t(u) = u e^{-t\nabla \chi_V(u)}$.

Rmk This is a gauge theoretic version of Teleman-Woodward's algebro-geometric result.

Rmk Similar for any number of boundary components & other Atiyah-Bott classes.

Proof outline

$$\operatorname{index}_\ell(E^\Sigma V) = \sum_{v\in \mathrm{Crit}(\|\mu\|^2)} \operatorname{index}\big(D_v + f_v\, c(\beta_v)\big)$$

Tian-Zhang / Braverman - type deformation of $D_v = D|_{U_v}$, with $f_v \to \infty$ at ∂U_v

Work of Braverman & Paradan - Vergne $\longrightarrow$ index formula for $\xi \in \mathfrak{t}$ small:

$$\operatorname{index}\big(D_v + f_v\, c(\beta_v)\big)(\exp \xi) = \int_{U_v} \hat{A}(U_v, \xi)\, \mathrm{Ch}(E^\Sigma V \otimes \mathcal{L}, \xi)\; P_v(\xi)\, \mathrm{Th}(\xi)$$

localizing form (generalized coeffs)

Non- abelian localization in ordinary cohomology (Paradan)

$$\operatorname{index}_\ell(E^\Sigma V)(\exp \xi) = \int_X \hat{A}(X, \xi)\, \mathrm{Ch}(E^\Sigma V \otimes \mathcal{L}, \xi)\; \mathrm{Th}(\xi)$$

Compute $\mathrm{Ch}(E^\Sigma V, \xi)$ (families index thm) & $\eta^* \mathrm{Ch}(E^\Sigma V, \xi) = \mathrm{Ch}(E^\Sigma V, \xi) + \nabla_\eta \chi_V(\exp \xi)$

$$\Rightarrow \quad \eta^* \mathrm{Ch}(e^{tE^\Sigma V} \otimes \mathcal{L}, \xi) = e^{\ell(\eta, \xi) + t \nabla_\eta \chi_V(\exp \xi)}\, \mathrm{Ch}(e^{tE^\Sigma V} \otimes \mathcal{L}, \xi)$$

$\int_X \square = \sum_{\eta \in \Pi} \int_{X_0} \eta^* \square$ Poisson summation. Evaluate resulting equiv. integral over X/Π.

Thanks!