Superconnection and family Bergman kernel

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Riemann-Roch-Grothendieck theorem

Family Bergman kernel

Idea of the proof

Dolbeault complex

- X compact complex manifold, $n = \dim X$.
- E a holomorphic vector bundle on X.
- $\overline{\partial}^E : \Omega^{0,q}(X,E) := \mathscr{C}^\infty(X, \Lambda^q(T^{*(0,1)}X) \otimes E) \to \Omega^{0,q+1}(X,E)$ the Dolbeault operator :

$$\overline{\partial}^E(\sum_j \alpha_j \xi_j) = \sum_j (\overline{\partial}\alpha_j)\xi_j.$$

ξ_j local holomorphic frame of E, and $\alpha_j \in \Omega^{0,q}(X)$.

$$(\overline{\partial}^E)^2 = 0.$$

Riemann-Roch-Hirzebruch theorem

- Dolbeault cohomology of X with values in E :

$$H^q(X,E) := H^{(0,q)}(X,E) := \frac{\ker(\overline{\partial}^E|_{\Omega^{0,q}})}{Im(\overline{\partial}^E|_{\Omega^{0,q-1}})}.$$

- Riemann-Roch-Hirzebruch theorem :

$$\sum_{q=0}^{n}(-1)^q \dim H^q(X,E) = \int_X \mathrm{Td}(T^{(1,0)}X)\,\mathrm{ch}(E).$$

Td( ) Todd class, ch( ) Chern class

Chern class

- h^E Hermitian metric on E, ∇^E Chern (holomorphic and Hermitian) connection on (E, h^E), its curvature

$$R^E = (\nabla^E)^2 \in \Omega^{(1,1)}(X, \mathrm{End}(E)).$$

-

$$\mathrm{Td}(E, h^E) = \det\left(\frac{R^E/2\pi i}{e^{R^E/2\pi i} - 1}\right),$$

$$\mathrm{ch}(E, h^E) = \mathrm{Tr}\left[e^{-R^E/2\pi i}\right].$$

They are closed forms in $\oplus_q \Omega^{(q,q)}(X)$.

- $\mathrm{Td}(E) := [\mathrm{Td}(E, h^E)], \mathrm{ch}(E) := [\mathrm{ch}(E, h^E)] \in \oplus_q H^q_{dR}(X)$ do not depend on h^E. $\mathrm{Td}(E)$ Todd class of E, $\mathrm{ch}(E)$ Chern class of E.

Riemann-Roch-Grothendieck theorem

- W, S compact complex manifolds.
  $\pi : W \to S$ holomorphic submersion with compact fiber X.
- E a holomorphic vector bundle on W.
- $H^\bullet(X, E|_X)$ fiberwise Dolbeault cohomology of X with values in E, forms the direct image $R^\bullet \pi_* E$ of E, a coherent sheaf on S.
  Ex : If $\dim H^\bullet(X, E|_X)$ is constant, then $H^q(X, E|_X)$ are holomorphic vector bundles on S.
- Riemann-Roch-Grothendieck theorem

$$\sum_{q=0}^{n} (-1)^q \operatorname{ch}(H^q(X, E|_X)) = \int_X \operatorname{Td}(T^{(1,0)}X) \operatorname{ch}(E) \in H^\bullet_{dR}(S).$$

Kodaira vanishing theorem

- L a holomorphic line bundle on W.
  E a holomorphic vector bundle on W.
- We suppose that L is positive along the fiber X, i.e., $\exists$ Hermitian metric h^L on L s.t. $\frac{i}{2\pi}R^L$ defines a Kähler form along X :

$$R^L(u,\overline{u}) > 0 \quad \forall 0 \neq u \in T^{(1,0)}X.$$

- Kodaira vanishing theorem : $\exists\ p_0 > 0$ s.t. $\forall p > p_0$,

$$H^q(X_b, L^p \otimes E|_{X_b})) = 0, \forall q > 0, b \in S.$$

Asymptotic R-R-G theorem

- Let $c_1(E) = \left[\frac{i}{2\pi}\operatorname{Tr}[R^E]\right] \in H^2_{dR}(X)$
- R-R-G Th : $\forall k > 0$, when $p \to +\infty$,

$$\{\operatorname{ch}(H^0(X, L^p \otimes E))\}^{(2k)} = \mathrm{rk}(E) \int_X \frac{c_1(L)^{n+k}}{(n+k)!} p^{n+k} + \int_X \Big(c_1(E) + \frac{\mathrm{rk}(E)}{2} c_1(T^{(1,0)}X)\Big) \frac{c_1(L)^{n+k-1}}{(n+k-1)!} p^{n+k-1} + \mathscr{O}(p^{n+k-2}) \in H^{2k}_{dR}(S).$$

degree $n+k$ polynomial on p.

- Question : Analytic refinement ?

Set up

- W, S compact complex manifolds. $\pi : W \to S$ holomorphic submersion with compact fiber X.
- (E, h^E) a hol. Herm. vector bundle on W. (L, h^L) a hol. Herm. line bundle on W s.t.
$$\omega = \frac{i}{2\pi} R^L$$
defines a Kähler form along X.
- L^2-metric $h^{H^0(X, L^p \otimes E)}$ on $H^0(X, L^p \otimes E)$
$$\langle s, s' \rangle = \int_X \langle s, s' \rangle (x) dv_X(x).$$
$dv_X = \frac{(\omega|_X)^n}{n!}$ Riemannian volume form along X.

Problem

- $$R_b^{H^0(X,L^p\otimes E)} = (\nabla^{H^0(X,L^p\otimes E)})^2 \in \Lambda^2(T^*_{\mathbb{R},b}S)\otimes \mathrm{End}(H^0(X_b, L^p\otimes E))$$

- Asymptotics of

$$(R^{H^0(X,L^p\otimes E)})^k \in \Lambda^{2k}(T^*_{\mathbb{R},b}S)\otimes \mathrm{End}(H^0(X_b, L^p\otimes E))?$$

How to formulate the problem ?
dim $H^0(X, L^p\otimes E)$ degree n polynomial on p !

Kernel of curvature

- $R_b^{H^0(X,L^p\otimes E)}(x,x')$ $(x,x'\in X_b, b\in S)$ smooth kernel of $R_b^{H^0(X,L^p\otimes E)}: H^0\to \Lambda^2\left(T^*_{\mathbb{R},b}S\right)\otimes H^0$ w. r. t. $dv_{X_b}(x')$.
- For $b\in S, x,x'\in X_b$,

$$R_b^{H^0(X,L^p\otimes E)}(x,x')\in \pi^*\left(\Lambda^2\left(T^*_{\mathbb{R},b}S\right)\right)\otimes (L^p\otimes E)_x\otimes (L^p\otimes E)^*_{x'}.$$

As $\mathrm{End}(L)=\mathbb{C}$, we get

$$R_b^{H^0(X,L^p\otimes E)}(x,x)\in \pi^*\left(\Lambda^2\left(T^*_{\mathbb{R},b}S\right)\right)\otimes \mathrm{End}\left(E_x\right).$$

Asymptotics of family Bergman kernel

- Theorem. $\exists\ b_{2,r} \in \mathscr{C}^\infty(W, \Lambda^2(T^*_{\mathbb{R}}S) \otimes \mathrm{End}(E))$ s.t. $\forall\ k, l \in \mathbb{N}$, $\exists$ $C_{k,\,l} > 0$ s. t. $\forall\ p \in \mathbb{N}$, $p > p_0$,

$$\left| R^{H^0(X, L^p \otimes E)}(x,x) - \sum_{r=0}^{k} b_{2,r}(x) p^{n-r+1} \right|_{\mathscr{C}^l(W)} \leqslant C_{k,\,l}\, p^{n-k},$$

- Δ_X is the (positive) Laplace operator of the fiber X_b,

$$\frac{\sqrt{-1}}{2\pi} b_{2,0} = \frac{(\omega^{n+1})^{(2)}}{(n+1)\,(\omega^n)^{(0)}}\,\mathrm{Id}_E = g^\alpha \wedge \overline{g}^\beta \omega\left(g^H_\alpha, \overline{g}^H_\beta\right)\mathrm{Id}_E,$$

$$\frac{\sqrt{-1}}{2\pi} b_{2,1} = \Bigg(\Big(\frac{1}{2} c_1(TX, h^{TX}) + \frac{\sqrt{-1}}{2\pi} R^E - \frac{1}{8\pi} g^\alpha \wedge \overline{g}^\beta \Delta_X\big(\omega(g^H_\alpha, \overline{g}^H_\beta)\big)\Big)\omega^n\Bigg)^{(2)} \Big/ (\omega^n)^{(0)},$$

Local refinement of Asymptotic R-R-G theorem

- $\forall k > 0$, when $p \to +\infty$,

$$
\begin{aligned}
& c_1(H^0(X, L^p \otimes E), \nabla^{H^0}) \\
& \qquad = \frac{i}{2\pi} \int_X \mathrm{Tr}[R^{H^0(X, L^p \otimes E)}(x,x)] dv_X(x) \\
& \qquad = \mathrm{rk}(E) \int_X \frac{c_1(L, \nabla^L)^{n+1}}{(n+1)!} p^{n+1} \\
& + \int_X \Big(c_1(E, \nabla^E) + \frac{\mathrm{rk}(E)}{2} c_1(T^{(1,0)}X, \nabla) \Big) \frac{c_1(L, \nabla^L)^n}{n!} p^n \\
& \qquad + \mathscr{O}(p^{n-1}) \in \Omega^2(S).
\end{aligned}
$$

Toeplitz operator I

- We fix $b \in S$. $P_p : \mathscr{C}^\infty(X, L^p \otimes E) \to H^0(X, L^p \otimes E)$ orthogonal projection. Bergman projection !
- For $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, Berezin-Toeplitz quantization of f :

$$T_{f,p} = P_p f P_p \in \mathrm{End}(H^0(X, L^p \otimes E)).$$

- A Toeplitz operator is a family of operators $\{T_p \in \mathrm{End}(H^0(X, L^p \otimes E))\}_{p \in \mathbb{N}^*}$ s. t. $\exists g_l \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ s.t. $\forall k \in \mathbb{N}, p \in \mathbb{N}^*$,

$$\left\| T_p - \sum_{l=0}^{k} p^{-l} T_{g_l,p} \right\| \leqslant C_k \, p^{-k-1}.$$

Berezin, Boutet de Monvel-Guillemin, Bordemann-Meinrenken-Schlichenmaier, Ma-Marinescu

Toeplitz operator II

- Ma-Marinescu : $\forall f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, $T_{f,p}\, T_{g,p}$ is a Toeplitz operator, and

$$T_{f,p}\, T_{g,p} = T_{fg,p} + T_{-\frac{1}{2\pi}\langle \nabla^{1,0} f, \overline{\partial}^E g\rangle_\omega,\, p}\, p^{-1} + \mathcal{O}(p^{-2}).$$

  based on Xianzhe Dai-Kefeng Liu-Ma's result on the off-diagonal asymptotics expansion of Bergman kernel $P_p(x, x')$.
- Thus Toeplitz operators form an algebra.

Geometric Quantization (Kostant, Souriau)

- Classical phase space :(X, ω)
  Quantum phase space $H^0(X, L)$
- Classical observables : Poisson algebra $\mathscr{C}^\infty(X)$,
  Quantum observables : linear operators on $H^0(X, L)$
- Semi-classical limit : $H^0(X, L^p)$, $p \to \infty$ is a way to relate the classical and quantum observables.

Curvature operator is a Toeplitz operator

- Theorem. $\frac{1}{p} R_b^{H^0(X,L^p\otimes E)}$ $(b \in S)$ is a Toeplitz operator : $\exists$ $g_{2,r} \in \mathscr{C}^\infty(W, \pi^*(\Lambda^2(T_{\mathbb{R}}^*S)) \otimes \mathrm{End}(E))$ s.t. $\forall k \in \mathbb{N}$,

$$R^{H^0(X,L^p\otimes E)} = \sum_{r=0}^{k} T_{g_{2,r},p}\, p^{-r+1} + \mathcal{O}(p^{-k}),$$

- $g_{2,0} = b_{2,0} = -2\pi i \frac{(\omega^{n+1})^{(2)}}{(n+1)\,(\omega^n)^{(0)}} \mathrm{Id}_E$, formula for $g_{2,1}$.
- Thus for $l \in \mathbb{N}$, $(\frac{1}{p} R^{H^0(X,L^p\otimes E)})^l$ is a Toeplitz operator and we compute the first two terms of the expansion.

Curvature operator is a Toeplitz operator

- Take $T^H_{\mathbb{R}} W = \{u \in T_{\mathbb{R}} W : \omega(u, X) = 0 \ \forall \ X \in T_{\mathbb{R}} X\}$, then

$$\omega = \omega^X + \omega^H \quad \text{with} \quad \omega^H = g^\alpha \wedge \overline{g}^\beta \omega(g^H_\alpha, \overline{g}^H_\beta).$$

We have

$$b_{2,0} = -2\pi\sqrt{-1}\omega^H,$$

$$b_{2,1} = \left(\left(\frac{1}{2}\operatorname{Tr}[R^{T^{(1,0)}X}] + R^E + \frac{\sqrt{-1}}{4}\Delta_X \omega^H\right)\omega^n\right)^{(2)} /(\omega^n)^{(0)},$$

$$g_{2,1} = \left(R^E + \frac{1}{2}\operatorname{Tr}[R^{T^{(1,0)}X}]\right)^H - \frac{\sqrt{-1}}{4}\Delta_X \omega^H.$$

Application

- We suppose $\omega = \frac{i}{2\pi} R^L$ defines a Kähler form on W.
- $(H^0(X, L^p \otimes E), h^{H^0})$ is Nakano positive for $p \gg 0$, more precisely, $\exists C, C_0 > 0$ s.t. $\forall p > 0$,

$$\dot{R}_p > (Cp - C_0)\,\mathrm{Id} \in \mathrm{End}(T^{(1,0)}S \otimes H^0(X, L^p \otimes E))$$

as Hermitian matrices.
Here for $u, v \in T_b^{(1,0)}S$, $\xi, \eta \in H_b^0$,

$$\langle R^{H^0}(u, \overline{v})\xi, \eta\rangle =: \langle \dot{R}_p(u \otimes \xi), v \otimes \eta\rangle.$$

Generalizations and applications

- (X, ω) compact symplectic manifold, $\dim_{\mathbb{R}} X = 2n$.
- (L, h^L) Hermitian (complex) line bundle on X.
- (E, h^E) Hermitian (complex) vector bundle on X.
- ∇^L, ∇^E Hermitian connections on (L, h^L), (E, h^E).
- Fundamental hypothesis :

$$\omega = c_1(L, \nabla^L) = \frac{\sqrt{-1}}{2\pi}(\nabla^L)^2.$$

- g^{TX} Riemannian metric on TX, J almost complex structure on TX, preserves ω and g^{TX}.
- Simplification : $g^{TX}(\cdot,\cdot) = \omega(\cdot, J\cdot)$.
  (Our results work without this assumption).

Dirac operator D_p on the spinors

- Symplectic case :
  Spinors : $\Lambda^{\cdot} := \Lambda^{\mathrm{even}}(T^{*(0,1)}X) \oplus \Lambda^{\mathrm{odd}}(T^{*(0,1)}X)$.
  $E_p := \Lambda^{\cdot} \otimes L^p \otimes E$.
  Dirac operator $D_p : \mathscr{C}^\infty(X, E_p^\pm) \longrightarrow \mathscr{C}^\infty(X, E_p^\mp)$,

$$D_p = \sum_i c(e_i) \nabla_{e_i}^{E_p}.$$

- Kähler case :

$$D_p := \sqrt{2}\left(\overline{\partial}^{L^p \otimes E} + \overline{\partial}^{L^p \otimes E,*}\right),$$

$$\ker D_p = H^\bullet(X, L^p \otimes E).$$

Bergman kernel

- P_p orthogonal projection from $\mathscr{C}^\infty(X, E_p)$ onto $\operatorname{Ker} D_p$.
- Bergman kernel : $P_p(x,x')$, $(x,x' \in X)$, is the $\mathscr{C}^\infty$ kernel of P_p associated to $\frac{\omega^n}{n!}(x')$.
- $P_p(x,x') \in (\Lambda^{\cdot} \otimes L^p \otimes E)_x \otimes (\Lambda^{\cdot} \otimes L^p \otimes E)^*_{x'}$.

$$P_p(x,x) \in \operatorname{End}(\Lambda^{\cdot}(T^{*(0,1)}X) \otimes E)_x.$$

  Dai-Liu-Ma : Asymptotic expansion for $P_p(x,x')$.
- A Toeplitz operator is a family $\{T_p\}$ of linear operators $T_p : L^2(X, E_p) \longrightarrow L^2(X, E_p)$ s. t.
  a)
$$T_p = P_p\, T_p\, P_p : \operatorname{Ker}(D_p) \to \operatorname{Ker}(D_p),$$
  b) $\exists\ g_l \in \mathscr{C}^\infty(X, \operatorname{End}(E))$ s.t.
  $\left\| T_p - \sum_{l=0}^{k} T_{g_l,p}\, p^{-l} \right\| \leqslant C_k\, p^{-k-1}$.

Curvature is a Toeplitz operator

- $T^H W$ be a sub-bundle of TW such that

$$TW = T^H W \oplus TX.$$

- For $U \in TS$, let $U^H \in T^H W$ s.t. $\pi_* U^H = U$, and

$$\nabla_U^{\ker D_p} \sigma = P_p \nabla_{U^H}^{E_p} P_p \sigma.$$

- Theorem.

$$R^{\ker D_p} = (\nabla^{\ker D_p})^2 \in \Lambda^2(T^* S) \otimes \mathrm{End}(\ker D_p)$$

is a Toeplitz operator.
It is useful in our work with Jean-Michel Bismut on the asymptotic of the analytic torsion.
Also Martin Puchol's work on the asymptotic of the holomorphic analytic torsion

Kähler case

- Kodaira embedding : X complex, $E = \mathbb{C}$, L holomorphic. For $p \gg 0$,

$$\Phi_p : X \hookrightarrow P(H^0(X, L^p)^*), \quad L^p = \Phi_p^* \mathcal{O}(1),$$
$$h^{L^p} = P_p(x,x)\ \Phi_p^* h^{\mathcal{O}(1)}.$$

  The asymptotic expansion of $P_p(x,x)$ when $p \to \infty$ was studied by Tian, Bouche, Ruan, Catlin, Zelditch, Lu, Xiaowei Wang, Dai-Liu-Ma

- Tian, Ruan : $\frac{1}{p}\Phi_p^*(\omega_{FS}) - \omega = \mathscr{O}(\frac{1}{p})$.
- Donaldson : If $Aut(X, L)$ is discrete. X has constant scalar curvature Kähler metric $\omega \in c_1(L)$, iff $\exists$ balance metric h_p on L^p for any $p \gg 1$ and $\frac{1}{p} c_1(L^p, h_p) \to \omega$.

Spectral gap

- Theorem (Ma, Marinescu (2002)) : for $p > 0$,

$$\mathrm{Spec}(\mathrm{D}_\mathrm{p}^2) \subset \{0\} \cup [4\pi \mathrm{p} - \mathrm{C_L}, +\infty[$$

Bismut-Vasserot $\cdots$

Superconnection

- For $\sigma \in \mathscr{C}^\infty(S, \Omega^{0,\bullet}(X, L^p \otimes E)) := \mathscr{C}^\infty(W, E_p)$,

$$\nabla^{\Omega}_U \sigma = \nabla^{E_p}_{U^H} \sigma \quad \text{for } U \in T_{\mathbb{R}} S.$$

B_p superconnection on $\Lambda(T^*_{\mathbb{R}} S) \widehat{\otimes} \Omega^{0,\bullet}(X, L^p \otimes E)$:

$$B_p = D_p + \nabla^{\Omega}.$$

- For $p \gg 1$, we have

$$R^{H^0(X, L^p \otimes E)} = \frac{1}{2\pi\sqrt{-1}} \left[\int_{|\lambda| = 2\pi p} \left(\lambda - B_p^2\right)^{-1} \lambda \, d\lambda \right]^{(2)}.$$

Along the base S, B_p is of order 1, but B_p^2 is of order 0 along the base S, and B_p^2 is a fiberwise second order elliptic operator, thus we can fix $b \in S$ and work fiberwisely as in the local family index theorem.

Superconnection

- $$(\lambda - B^2)^{-1} = (\lambda - D^2)^{-1} + (\lambda - D^2)^{-1} \sum_{j=1}^{\dim_{\mathbb{R}} S} \Big((B^2)^{(>0)} (\lambda - D^2)^{-1} \Big)^j,$$
  $$P(B^2)^{(1)} P = 0.$$
- Spectral gap implies

$$\begin{aligned}
R^{\mathrm{Ker}(D_p^X)} &= P_p (\nabla^{\Omega})^2 P_p - P_p [\nabla_p^{\Omega}, P_p] P_p^{\perp} [\nabla_p^{\Omega}, P_p] P_p \\
&= P_p (\nabla^{\Omega})^2 P_p - P (B_p^2)^{(1)} ((B_p^2)^{(0)})^{-1} P_p^{\perp} (B_p^2)^{(1)} P_p \\
&= \frac{1}{2\pi\sqrt{-1}} \left[\int_{|\lambda| = 2\pi p} (\lambda - B_p^2)^{-1} \lambda d\lambda \right]^{(2)} \\
&= \frac{p}{2\pi\sqrt{-1}} \left[\int_{|\lambda| = 2\pi} \left(\lambda - \frac{1}{p} B_p^2 \right)^{-1} \lambda d\lambda \right]^{(2)} .
\end{aligned}$$

Local family index theorem

- Bismut superconnection on $\Lambda(T^*_{\mathbb{R}}S)\widehat{\otimes}\Omega^{0,\bullet}(X,E)$:

$$A_t = \sqrt{t}D_0 + \nabla^{\Omega} + \frac{1}{4\sqrt{t}}c(T),$$

with $T(U,V) := -P^{TX}[U^H, V^H]$ for $U, V \in TS$.

- Theorem (Bismut 1986). $\mathrm{Tr}_s[e^{-A_t^2}]$ is a closed form, and its cohomology class does not depend on t, and is given by $\mathrm{ch}(\ker D_0)$. Finally

$$\lim_{t\to 0} \mathrm{Tr}_s[e^{-A_t^2}] = \int_X \mathrm{Td}(T^{(1,0)}X, \nabla^{TX})\,\mathrm{ch}(E, \nabla^E).$$

- $\Rightarrow$ Atiyah-Singer family index theorem :

$$\mathrm{ch}(\ker D_0) = \int_X \mathrm{Td}(T^{(1,0)}X)\,\mathrm{ch}(E) \in H^{\bullet}_{dR}(S).$$