Delocalized invariants for proper actions of Lie groups

Based on joint work with P. Piazza, Y. Song and X. Tang

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Motivation: Numerical invariants from index classes

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D invariant Dirac operator $\Longrightarrow \mathrm{Ind}(D) \in K_0(C_r^*(\Gamma))$
If τ_φ extends to a subalgebra $\mathbb{C}[\Gamma] \subset \mathcal{A} \subset C_r^*(\Gamma)$ we can pair:

$$\langle \mathrm{Ind}(D), \tau_\varphi \rangle = \int_{M/\Gamma} AS(D) \wedge \nu^* \varphi.$$

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- **Secondary invariants**: higher APS index theorem (Leichtnam–Piazza,...). Assume D_∂ is L^2-invertible:

$$\langle \mathrm{Ind}(D), \tau_\varphi \rangle = \int_{M/\Gamma} AS(D) \wedge \nu^* \varphi - \frac{1}{2}\eta_\varphi(D_\partial).$$

$\eta_\varphi(D_\partial)$ *higher* η*-invariant.*

$$\eta_0(D) := \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{Tr}\left(De^{-tD^2}\right) \frac{dt}{\sqrt{t}}$$

$\Rightarrow$ higher ρ-invariants.

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- Instead of discrete groups consider Lie groups (connected, real reductive)
- Allow for general proper actions, instead of free actions
- Consider *delocalized cyclic cocycles*.
- N.B. $HC^\bullet(C_c^\infty(G))$ is a module over

$$C^\infty_{\rm inv}(G) := \{f \in C^\infty(G),\ f(hgh^{-1} = f(g)\}.$$

  A cocycle is called *delocalized* if its image in $HC^\bullet(C_c^\infty(G))_{m_e}$ is zero, where $m_e := \{f,\ f(e) = 0\}$.
- The Plancherel trace is *localized at the identity*:

$$\tau_e(f) := f(e).$$

Paolo's questions

- can we define a delocalized eta invariant $\eta_g(D)$ using the heat kernel?
- if C is a smoothing perturbation can we define $\eta_g(D + C)$?
- in particular, can we define the delocalized eta invariant of a PSC metric g and of a G-equivariant homot. equivalence f ?
- is there a delocalized APS index theorem ?
- are there higher versions of these results ?

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The answer to all these questions is: Yes!

Outline

Delocalized η-invariants

APS index theorem

Perturbations

Higher versions

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- G linear real reductive, connected Lie group, K maximal compact subgroup. Equal rank: $\dim(G/K) =$ even.

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$$\nu_t(f) := \int_G (1 + \|g\|)^t \Xi^{-1}(g) |f(g)| dg,$$

  - $g \mapsto \|g\|$ riemannian distance from eK to gK in G/K.
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- Let $g \in G$ semisimple. Orbital integral

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- N.B. Other versions: Harish–Chandra algebra $\mathcal{C}(G)$, rapid decay $H_L^\infty(G)$.

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  - **Spin^c-structure**: K-invariant Spin^c-structures on S and $\mathfrak{p}$
- **Dirac operator**

$$D = D_{G,K} \hat{\otimes} 1 + 1 \hat{\otimes} D_S,$$

$D_{G,K}$ Spin^c-Dirac operator on $L^2(G) \otimes S_{\mathfrak{p}}$.

Delocalized traces

- Subalgebras of the Roe algebra $C^*(X)^G$:

$$\mathcal{A}^c_G(X) := (C^\infty_c(G)\hat{\otimes}\Psi^{-\infty}(S))^{K\times K}$$
$$\cong \{\Phi : G \to \Psi^{-\infty}(S),\ \text{compact support},\ K\times K \text{ invariant.}\}$$

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$$\tau_g^X(\Phi) := \tau_g(\mathrm{Tr}_S(\Phi)).$$

Delocalized η-invariants

X is of bounded geometry and $\mathcal{A}^\infty_G(X) \subset C^*(X)^G$ is closed under holomorphic functional calculus:

$$e^{-tD^2} = \frac{1}{2\pi i}\int_C \frac{e^{-t\lambda}}{\lambda - D^2}d\lambda \in \mathcal{A}^\infty_G(X).$$

Follows from analysis of the resolvent

$$\lambda - D^2 = B_\lambda + C_\lambda, \qquad B_\lambda \in \Psi_c^{-2}(X),\ C_\lambda \in \mathcal{A}^\infty_G(M),$$

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Theorem (Piazza–P.–Song–Tang)

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N.B: No invertibility or gap assumptions!

About the proof

Large time behaviour: Recall $D^2 = D_{G,K}^2 + D_S^2$ and decompose

$$L^2(X) = \bigoplus_{\lambda_i \in \sigma(D_S)} \left[L^2(G) \otimes \mathfrak{p} \otimes L^2(S)_{\lambda_i} \right]^K .$$

Moscovici–Stanton give estimates
$|\tau_g^{G/K}(D_{G,K} e^{-D_{G,K}^2})| \leq C_1 e^{-C_2/t} t^{-3/2}$ resulting in convergence of

$$\frac{1}{\sqrt{\pi}} \int_1^\infty \tau_g^X (De^{-tD^2}) \frac{dt}{\sqrt{t}}.$$

About the proof

Short time behaviour: rewrite (Hochs–Wang)

$$\tau_g^X(De^{-tD^2}) = \int_X \int_G c_G(g) c_X(gx) \mathrm{tr}(k_t(x, gx)) dg dx,$$

with $k_t(x, y)$ kernel of De^{-tD^2}, and c^X cut-off function on X.

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Let $W \subset X$ open subset containing X^g.

- Gaussian estimates for the heat kernel gives exponential decay of

$$\int_{X \setminus W} \int_G c_G(g) c_X(gx) \mathrm{tr}(k_t(x, gx)) dg dx, \quad \text{as } t \downarrow 0.$$

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- An argument of Zhang shows that

$$\frac{1}{\sqrt{t}} \int_W \int_G c_G(g) c_X(gx) \mathrm{tr}(k_t(x, gx)) dg dx = O(1) \quad t \downarrow 0.$$

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Using b-calculus with ϵ-bounds on the slice $S \subset Y$:

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Connes–Skandalis projector
$(D^+ Q^b = 1 - {}^b S_-, \ Q^b D^{-1} = 1 - {}^b S_+)$

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gives an index class

$$\mathrm{Ind}_\infty(D) := [P_Q^b] - [e_1] \in K_0(\mathcal{A}_G^\infty(Y)) = K_0(C^*(Y_0 \subset Y)^G)$$

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$$V(D) := \begin{pmatrix} e^{-D^- D^+} & e^{-\frac{1}{2}D^- D^+} \left(\frac{I - e^{-D^- D^+}}{D^- D^+} \right) D^- \\ e^{-\frac{1}{2}D^+ D^-} D^+ & I - e^{-D^+ D^-} \end{pmatrix} \in M_2({}^b\mathcal{A}^\infty_G(Y))$$

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Relative index class

$\mathrm{Ind}_\infty(D, D_\partial) \in K_0({}^b\mathcal{A}^\infty_G(Y_0), {}^b\mathcal{A}^\infty_{G,\mathbb{R}}(\mathrm{cyl}(\partial Y)))$:

$$(V(D), e_1, q_t)\,, \quad t \in [1, +\infty]\,, \quad \text{with } q_t := \begin{cases} V(tD_{\rm cyl}) & \text{if} \quad t \in [1, +\infty) \\ e_1 & \text{if} \quad t = \infty \end{cases}$$

Relative cyclic cocycles

- Recall the delocalized trace:

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- Using the b-trace $\mathrm{Tr}_S^b(k) := {}^b\!\!\int_S k(x,x)dx$,

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- 1-cocycle on ${}^b\mathcal{A}_{G,\mathbb{R}}^\infty(\mathrm{cyl}(\partial Y))$:

$$\sigma_g^{\partial Y}(A_0, A_1) := \frac{i}{2\pi}\int_{\mathbb{R}} \tau_g^{\partial Y}(\partial_\lambda I(A_0,\lambda) \circ I(A_1,\lambda) d\lambda,$$

where the indicial family of $A \in {}^b\mathcal{A}_{G,\mathbb{R}}^\infty(\mathrm{cyl}(\partial Y))$, denoted $I(A,\lambda)$, appears.

Index pairings

- Recall Melrose formula for the b-trace:

$$ {}^b\mathrm{Tr}_S([A_0, A_1]) = \frac{i}{2\pi} \int_{\mathbb{R}} \mathrm{Tr}_{\partial S}(\partial_\lambda(A_0, \lambda) \circ I(A_1, \lambda)) d\lambda $$

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- This implies

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- and leads to

$$ \langle \tau_g^Y, \mathrm{Ind}_\infty(D) \rangle = \langle (\tau_g^{Y,r}, \sigma_g), \mathrm{Ind}_\infty(D, D_\partial) \rangle. $$

The delocalized APS Theorem

Theorem (Piazza–P.–Song–Tang)

Assume that D_∂ is L^2-invertible.

$$\langle \tau_g^Y, \mathrm{Ind}_\infty(D) \rangle = \int_{Y_0^g} c^g \mathrm{AS}_g(D_0) - \frac{1}{2}\eta_g(D_{\partial Y}),$$

where

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- Closed manifold case is due to Hochs–Wang.

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- Pairing on the boundary with $\sigma_g^{\partial Y}$ produces a long complicated expression that we show that this equals $\eta_g(D)$.

Three types of perturbations

For the signature operator D_∂ is not invertible.

What if D_∂ is not L^2-invertible?

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- general perturbation $D + C$ with $C \in \mathcal{A}^c_G(X)$. (always exists!)

Global perturbation on the slice

- Recall $D = D_{G,K} \otimes 1 + \gamma \otimes D_S$. The Connes–Kasparov correspondence gives

$$D \; L^2 - \text{invertible} \quad \Longleftrightarrow \quad D_S \; L^2 - \text{invertible}$$

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Theorem (Piazza–P.–Song–Tang)

Assume G is of equal rank, i.e., has discrete series. Then

$$\left\langle \mathrm{Ind}_\infty(D_\vartheta), \tau_g^Y \right\rangle = \int_{Y_0^g} c^g \mathrm{AS}_g(D_0) - \frac{1}{2} \left(\eta_g(D_{\partial Y}) + \langle \mathrm{Ind}_D(\ker(D_S)), \tau_g \rangle \right)$$

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Theorem (Piazza–P.–Song–Tang)

Assume that D has a spectral gap.

$$\left\langle \mathrm{Ind}_\infty(D_\vartheta), \tau_g^Y \right\rangle = \int_{Y_0^g} c^g \mathrm{AS}_g(D_0) - \frac{1}{2} \left(\eta_g(D_{\partial Y}) + \left\langle \Pi_{\ker(D_\partial)}, \tau_g \right\rangle \right)$$

General perturbations

- Previous cases work under a gap assumption. We now consider

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Higher versions

- Song–Tang: For any cuspidal parabolic subgroup $P = MAN \subset G$ and $g \in G$ semisimple there exists a cyclic cocycle Φ_g^P of degree $\dim(A)$ on $\mathcal{L}_t(G)$.

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- $\Rightarrow$ cyclic cocycle $\Phi_{Y,g}^P$ on $\mathcal{A}_G^\infty(Y)$.
- Higher APS index theorem:

Theorem (Piazza–P.–Song–Tang)

Assume that D_∂ is L^2-invertible.

$$\left\langle \Phi_{Y,g}^P, \mathrm{Ind}_\infty(D) \right\rangle = \int_{(Y_0/AN)^g} c_{Y_0/AN}^g AS(Y_0/AN)_g - \frac{1}{2}\eta_g(D_{\partial Y_M}),$$

where $Y_M := M \times_{K \cap M} S$ is the slice decomposition of Y/AM with its M-action.

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- If $g \in G$ is not elliptic, any element of its conjugacy class C_g will act fixed point free on any proper G-manifold W:

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- $\mathbf{f} : X_1 \to X_2$ G-homotopy invariant. Fukumoto: there exists a perturbation $B_{\mathbf{f}}$ making the signature operator on $X := X_1 \sqcup (-X_2)$:

$$\rho_g(\mathbf{f}) := \eta_g(D_X^{\mathrm{sign}} + B_{\mathbf{f}}).$$

  Bordism invariant by the APS index theorem.

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