Cyclic cohomology and Topological K-theory for discrete groups

The Chern-Baum-Connes assembly map

P. Carrillo Rouse[1]

[1]Institut de Mathématiques de Toulouse

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Goal of the talk:

Given a discrete countable group Γ, Baum and Connes defined an assembly map

$$K^*_{top}(\Gamma) \xrightarrow{\mu_\Gamma} K_*(C^*_r(\Gamma))$$

by assembling higher indices of Γ-invariant elliptic operators. The main goal of the talk is to present

1. A Chern-Baum-connes assembly map

$$\begin{array}{ccc} K^*_{top}(\Gamma) & \xrightarrow{\mu_\Gamma} & K_*(C^*_r(\Gamma)) \\ & \searrow^{Ch_\Gamma} & \vdots \downarrow \\ & & HP_*(\mathbb{C}\Gamma) \end{array}$$

2. A geometric formula for the induced pairing with $HP^*(\mathbb{C}\Gamma)$.

Classic Atiyah-Singer Index theory and wrongway functoriality

For M closed smooth mfd

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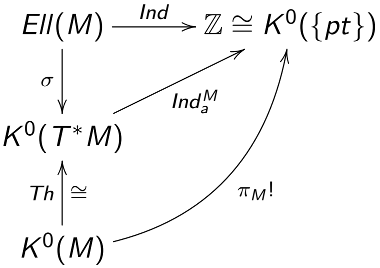

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Once giving sense to $f! : K^*(M) \to K^*(M')$ for appropriate $f : M \to M'$, the Index theorem is equivalent to pushforward functoriality

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And the topological Index formula by comparing these shrieks maps $f!$ with their cohomological version! (RR theorem)

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And again the topological Index formula by comparing these shrieks maps $f!$ with their cohomological version! (RR theorem)

Baum-Connes assembly map

One can consider more involved spaces at the place of B, for instance a discrete group Γ!!!!

Baum-Connes assembly map

Let Γ be a discrete (countable) group.
For M a $\Gamma-$proper *spin*^{c} manifold Baum and Connes were originally interested in higher index maps

$$K_\Gamma^*(M) \xrightarrow{\pi_M!} K_*(C_r^*(\Gamma)),$$

and higher index formulas (as for example the higher signatures of Novikov) and applications (for example towards the homotopy invariance of the higher signatures)

In the original papers of Baum and Connes, they assembled such maps using pushforward maps in K-theory:
Given $f : M \to N$ a Γ, K-oriented map, they wanted first

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to get an assembly map

$$\mathrm{K}^*_{\mathrm{top}}(\Gamma) \xrightarrow{\mu} \mathrm{K}_*(\mathrm{C}^*_{\mathrm{r}}(\Gamma))$$

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where

$$K^*_{top}(\Gamma) := \varinjlim_{f!} K^*_\Gamma(M)$$

and $\mu([M, x]) := \pi_M!(x)$.

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- Through the years other models were proposed for the left hand side, Baum-Douglas, Luck.
- But what about the original model?

In collaboration with B.L. Wang (AENS 2014) we used deformation groupoids to define

$$K^*_\Gamma(M) \xrightarrow{f!} K^*_\Gamma(N),$$

for $f : M \to N$ "good" Γ-map (in particular for $N = \{pt\}$!). And we proved the functoriality

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!

The importance of the original model is that it was developed to compute explicit Higher Index Formulas!!!

Digression on Deformation to the normal cone and pushforward maps!

For $X \subset M$ submanifold in a manifold, there is a mfd, called the Deformation to the normal cone (DNC)

$$D(M,X) := N(M,X) \times \{0\} \bigsqcup M \times \mathbb{R}^*.$$

Main feature: This construction has good functorial properties: For $f : (M,X) \to (M',X')$ there is a C^∞-map

$$D(f) : D(M,X) \to D(M',X')$$

with $D(f \circ g) = D(f) \circ D(g)$, $D(Id) = Id$,...

Corollary

For a groupoid inclusion (immersion more generally) $G_1 \hookrightarrow G_2$ the DNC construction gives a Lie groupoid

$$D(G_2, G_1) \rightrightarrows D(G_2^{(0)}, G_1^{(0)}).$$

Example: For $M \hookrightarrow M \times M$ on gets Connes Tangent groupoid

$$G_M^{tan} := D(M \times M, M) \rightrightarrows M \times [0, 1].$$

Recall construction of the pushforward $f!$ in equivariant K-theory:

- For $f : M \to N$ as above consider the groupoid immersion

$$M \xrightarrow{f \times \triangle} N \times (M \times M)$$

and the DNC **LIE** groupoid

$$D_f := (TM \oplus f^* TN) \times \{0\} \bigsqcup (M \times M) \times N \times (0,1] \rightrightarrows f^* TN \times \{0\} \bigsqcup M \times N \times (0,1]$$

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$$0 \longrightarrow C^*((M \times M) \times N \times (0,1]) \longrightarrow C^*(D_f) \xrightarrow{e_0} C^*(TM \oplus f^* TN) \longrightarrow 0$$

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- Applying K-theory to it one gets $f!$ as:

$$K^*(M) \xrightarrow[\cong]{Th} K^*(T^*M \oplus f^* T^*N)) \xrightarrow[\cong]{F} K^*(TM \oplus f^* TN)) \xrightarrow[\cong]{(e_0)_*^{-1}}$$

$$K^*(D_f) \xrightarrow{(e_1)_*} K^*((M \times M) \times N) \xrightarrow[\cong]{M} K^*(N)$$

In the equivariant case, thanks to the fuctoriality of the Deformation to the Normal Cone construction we get

- A semi-direct product groupoid

$$D_f \rtimes \Gamma \rightrightarrows f^*TN \times \{0\} \bigsqcup M \times N \times (0,1]$$

  and

- the pushforward $f!$

$$K^*_\Gamma(M) \xrightarrow[\cong]{Th} K^*_\Gamma(T^*M \oplus f^*T^*N)) \xrightarrow[\cong]{F} K^*_\Gamma(TM \oplus f^*TN)) \xrightarrow[\cong]{(e_0)_*^{-1}}$$

$$K^*_\Gamma(D_f) \xrightarrow{(e_1)_*} K^*_\Gamma((M \times M) \times N) \xrightarrow[\cong]{M} K^*_\Gamma(N)$$

Pushforward and Riemann-Roch theorem in delocalized cohomology

For M a $\Gamma-$proper manifold there are groups

$$H^*_{\Gamma,deloc}(M) := \bigoplus_{\langle\Gamma\rangle^{fin}} \prod_{k=*,\, mod\, 2} H^k_c(M_g \rtimes \Gamma_g),$$

even and odd, that might be also defined by periodizing the de Rham cohomology of the so called inertia groupoid.

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Why these are the good groups?

Because there is a (Tu and Xu) Γ-Chern character morphism and isomorphism up to tensoring with $\mathbb{C}$:

$$ch_M^\Gamma : K_\Gamma^*(M) \longrightarrow H^*_{\Gamma,deloc}(M)$$

given explicitly as the composition of the Connes-Chern character

$$Ch : K_\Gamma^*(M) \longrightarrow HP_*(C_c^\infty(M \rtimes \Gamma)).$$

followed by an isomorphism

$$HP_*(C_c^\infty(M \rtimes \Gamma)) \xrightarrow[\cong]{TX} H^*_{\Gamma,deloc}(M).$$

explicitly described by Tu and Xu.

Pushforward in delocalized cohomology

Let $f : M \longrightarrow N$ be a Γ, K-oriented C^∞-map between two $\Gamma-$proper cocompact manifolds. We define the morphism

$$f_! : H^{*-r_f}_{\Gamma,deloc}(M) \longrightarrow H^*_{\Gamma,deloc}(N).$$

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- the isomorphism

$$F : HP_*(\mathscr{S}(T_f^* \rtimes \Gamma)) \longrightarrow HP_*(\mathscr{S}(T_f \rtimes \Gamma))$$

  induced from the Fourier algebra isomorphism,

- the deformation morphism

$$I_f : HP_*(\mathscr{S}(T_f \rtimes \Gamma)) \longrightarrow HP_*(C_c^\infty((M \times M \times N) \rtimes \Gamma)),$$

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$$\mathcal{M} : HP_*(C_c^\infty((M \times M \times N) \rtimes \Gamma)) \longrightarrow HP_*(C_c^\infty(N \rtimes \Gamma)),$$

and,

- the deformation morphism

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  and,

- the Tu-Xu isomorphism

$$TX : HP_*(C_c^\infty(N \rtimes \Gamma)) \longrightarrow H^*_{\Gamma,deloc}(N)$$

delocalized Riemann-Roch

Theorem (CR+Wang+Wang, 2020)

Let $f : M \longrightarrow N$ be a Γ, K-oriented C^∞-map between two $\Gamma-$proper cocompact manifolds M and N. The following diagram is commutative

$$\begin{array}{ccc} K_\Gamma^{*-r_f}(M) & \xrightarrow{f!} & K_\Gamma^*(N) \\ \downarrow{\scriptstyle ch_{Td_M^\Gamma}} & & \downarrow{\scriptstyle ch_{Td_N^\Gamma}} \\ H_{\Gamma,deloc}^{*-r_f}(M) & \xrightarrow[f!]{} & H_{\Gamma,deloc}^*(N) \end{array}$$

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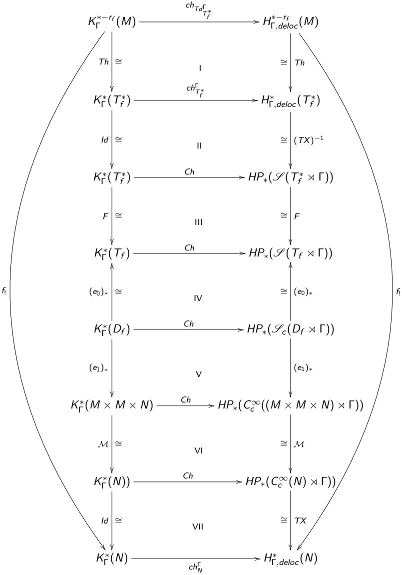
Where $ch_{Td_M^\Gamma}$ is the morphism

$$ch_{Td_M^\Gamma} : K_\Gamma^*(M) \longrightarrow H_{\Gamma,deloc}^*(M)$$

given by $x \mapsto ch_M^\Gamma(x) \wedge Td_M^\Gamma$, where Td_M^Γ is a delocalized Todd class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Th $\cong$
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$H^*_{\Gamma,deloc}(N)$
ch^{Γ}_N

Wrong way functoriality in delocalized cohomology and cohomological assembly

Theorem (CR+Wang+Wang, 2020)

Let $f : M \longrightarrow N$ and $g : N \longrightarrow L$ be Γ, K-oriented C^∞-maps between $\Gamma-$proper cocompact manifolds as above. Then

$$(g \circ f)! = g! \circ f!$$

as morphisms from $H^*_{\Gamma,deloc}(M)$ to $H^{*+r_{g\circ f}}_{\Gamma,deloc}(L)$ ($r_{g\circ f} = r_f + r_g \mod 2$).

Wrong way functoriality in delocalized cohomology and cohomological assembly

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as morphisms from $H^*_{\Gamma,deloc}(M)$ to $H^{*+r_{g\circ f}}_{\Gamma,deloc}(L)$ ($r_{g\circ f} = r_f + r_g \mod 2$).

The proof follows the same lines of the K-theoretical proof using deformation groupoids (CR+Wang AENS) using exactly the same groupoids but this time using appropriate Schwartz algebras!!!

delocalized topological cohomology for discrete groups

With the pushforward functoriality we can assemble the groups $H^*_{\Gamma,deloc}(M)$.
To be precise, for a discrete group Γ and for $* = 0, 1 \; mod \, 2$ we can consider the abelian group

$$H^*_{top}(\Gamma) = \varinjlim_{f!} H^*_{\Gamma,deloc}(M).$$

where the limit is taken over the Γ proper co-compact $spin^c$ manifolds of dimension equal to $*$ modulo 2.

We can start by assembling the Tu-Xu Chern characters, that is, thanks to the delocalized Riemann-Roch theorem and the delocalized wrong way functoriality we have

Theorem (Chern character for discrete groups)

For any discrete group Γ there is a well defined Chern character morphism

$$ch^{top} : K^{top}(\Gamma) \longrightarrow H^{*}_{top}(\Gamma)$$

given by

$$ch^{top}([M, x]) = [M, ch^{\Gamma}_{M}(x) \wedge Td^{\Gamma}_{M}].$$

Even more, it is an isomorphism once tensoring with $\mathbb{C}$.

Cohomological assembly map

For M a Γ-proper manifold as above we can define

$$\pi_M! : H^*_{\Gamma,deloc}(M) \to H_*(\Gamma, F\Gamma)$$

where

$$H_*(\Gamma, F\Gamma) := \left(\bigoplus_{\langle\Gamma\rangle^{fin}} \bigoplus_{k=*,\, mod\, 2} H_k(\Gamma_g; \mathbb{C}) \right)$$

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Idea: For $\omega \in \Omega^n_c(M \times \Gamma^p)$ and $p \in \mathbb{N}$. One can show $\pi_M!(\omega) \in \mathbb{C}\Gamma^p$ given by

$$\pi_M!(\omega)(\gamma) := \int_M \omega(m, \gamma)$$

for $\gamma \in \Gamma^p$, induces the morphism $\pi_M!$ above!

Theorem (Cohomological assembly in group homology)

The maps

$$H^*_{\Gamma,deloc}(M) \xrightarrow{\pi_M!} H_*(\Gamma, F\Gamma),$$

induce a well defined cohomological assembly map

$$\mu_{F\Gamma} : H^*_{top}(\Gamma) \longrightarrow H_*(\Gamma, F\Gamma)$$

given then by

$$\mu_{F\Gamma}([M,\omega]) = \pi_M!(\omega).$$

*(**NEW!**) Moreover, $\mu_{F\Gamma}$ is an isomorphism.*

Strategy of the proof:

$$\begin{array}{c} H^*_{top}(\Gamma) \\ \Big\downarrow \mu_{F\Gamma} \\ H_*(\Gamma, F\Gamma) \end{array}$$

Strategy of the proof:

$$\begin{array}{c} K^*_{top}(\Gamma) \otimes \mathbb{C} \\ \cong \Big\downarrow ch_{top} \\ H^*_{top}(\Gamma) \\ \Big\downarrow \mu_{F\Gamma} \\ H_*(\Gamma, F\Gamma) \end{array}$$

Strategy of the proof:

$$
\begin{array}{c}
K_*^\Gamma(\underline{E\Gamma}) \otimes \mathbb{C} \\
\cong \Big\uparrow \lambda \\
K^*_{top}(\Gamma) \otimes \mathbb{C} \\
\cong \Big\downarrow ch_{top} \\
H^*_{top}(\Gamma) \\
\Big\downarrow \mu_{F\Gamma} \\
H_*(\Gamma, F\Gamma)
\end{array}
$$

Strategy of the proof:

$$
\begin{array}{ccc}
\bigoplus_{g \in <\Gamma>^{fin}} K_*(B\Gamma_g) \otimes \mathbb{C} & \xrightarrow[\cong]{\phi_*^\Gamma} & K_*^\Gamma(\underline{E\Gamma}) \otimes \mathbb{C} \\
 & & \cong \big\uparrow \lambda \\
 & & K^*_{top}(\Gamma) \otimes \mathbb{C} \\
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Strategy of the proof:

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\begin{array}{ccc}
\bigoplus_{g\in<\Gamma>^{fin}} K_*(B\Gamma_g)\otimes\mathbb{C} & \xrightarrow[\cong]{\phi_*^\Gamma} & K_*^\Gamma(\underline{E\Gamma})\otimes\mathbb{C} \\
\lambda \Big\uparrow \cong & & \cong \Big\uparrow \lambda \\
\bigoplus_{g\in<\Gamma>^{fin}} K^*_{topf}(\Gamma_g)\otimes\mathbb{C} & \xrightarrow[\phi_*^\Gamma]{\cong} & K^*_{top}(\Gamma)\otimes\mathbb{C} \\
 & & \cong \Big\downarrow ch_{top} \\
 & & H^*_{top}(\Gamma) \\
 & & \Big\downarrow \mu_{F\Gamma} \\
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\end{array}
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Back to cyclic theory for final statement

In fact by Burghelea's work there is a morphism

$$H_*(\Gamma, F\Gamma) \xrightarrow{B} HP_*(\mathbb{C}\Gamma)$$

which is an isomorphism onto its image as a direct factor. In particular we can consider the assembly map

$$\mu_{top} : H^*_{top}(\Gamma) \longrightarrow HP_*(\mathbb{C}\Gamma)$$

given by the composition of the cohomological assembly map $\mu_{F\Gamma}$ of the theorem above followed by Burghelea's morphism B.

Back to cyclic theory for final statement

In fact, there is a commutative diagram

$$\begin{array}{ccc} H^*_{top}(\Gamma) & \xrightarrow{\mu_{F\Gamma}} & H_*(\Gamma, F\Gamma) \\ \lambda \downarrow \cong & & \downarrow B \\ H^\Gamma_*(\underline{E\Gamma}) & \xrightarrow[\mu]{} & HP_*(\mathbb{C}\Gamma) \end{array}$$

where the bottom assembly μ is the algebraic assembly associated to $HP_*(\mathbb{C}\cdot)$ which, by Cortinas-Tartaglia, is an isomorphism into its image, the factor that corresponds to $H_*(\Gamma, F\Gamma)$ via the Burghelea morphism.

Burghelea and other authors have also computed

$$HP^*(\mathbb{C}\Gamma) \cong \left(\prod_{\langle\Gamma\rangle^{fin}} \bigoplus_{k=*,\, mod\, 2} H^k(\Gamma_g;\mathbb{C}) \right) \bigoplus \prod_{\langle\Gamma\rangle^{\infty}} T^*(g,\mathbb{C})$$

We also need to notice before the final statement that there are canonical morphisms (for every g of finite order)

$$H^\bullet(\Gamma_g;\mathbb{C}) \xrightarrow{\pi_g^*} H^\bullet(M_g \rtimes \Gamma_g)$$

induced from the canonical groupoid projections $M_g \rtimes \Gamma_g \to \Gamma_g$.

Theorem (The Chern-Baum-Connes assembly map)

For any discrete countable group Γ*, there is well defined morphism*

$$K^*_{top}(\Gamma) \xrightarrow{ch^{top}_{\mu}} HP_*(\mathbb{C}\Gamma)$$

given by the composition

$$K^*_{top}(\Gamma) \xrightarrow{ch_{top}} H^*_{top}(\Gamma) \xrightarrow{\mu_{top}} HP_*(\mathbb{C}\Gamma)$$

inducing a well defined pairing

$$K^*_{top}(\Gamma) \times HP^*(\mathbb{C}\Gamma) \to \mathbb{C}$$

computed in every g*-component (with respect to Burghelea's decompostion above) by*

$$\langle [M, x], \tau_g \rangle = \langle ch^g_M(x) \wedge Td^M_g, \pi^*_g(\tau_g) \rangle$$

In the formula

$$\langle [M,x],\tau_g\rangle = \langle ch_M^g(x)\wedge Td_g^M, \pi_g^*(\tau_g)\rangle$$

the right hand side pairing corresponds to the pairing between Γ_g-invariant forms and currents on M_g.

We call the morphism ch_μ^{top} above the Chern-Baum-Connes assembly map of the group Γ.

An orbifold refinement for the BC and the CBC assembly maps: Work in progress with B.L. WANG and H. WANG, and with VELASQUEZ

Using deformation groupoids as above one can consider

1.
2.

An orbifold refinement for the BC and the CBC assembly maps: Work in progress with B.L. WANG and H. WANG, and with VELASQUEZ

Using deformation groupoids as above one can consider

1. An Orbifold topological K-theory group $K^*_{top,orb}(\Gamma)$ with orbifold cycles instead of only smooth manifold cycles.
2.

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Using deformation groupoids as above one can consider

1. An Orbifold topological K-theory group $K^*_{top,orb}(\Gamma)$ with orbifold cycles instead of only smooth manifold cycles.
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One can show that there is a commutative diagram

$$\begin{array}{ccc} K^*_{top,orb}(\Gamma) & \xrightarrow{\mu_\Gamma^{orb}} & K^*(\Gamma) \\ \uparrow{\scriptstyle\lambda} & \nearrow{\scriptstyle\mu_\Gamma} & \\ K^*_{top}(\Gamma) & & \end{array} .$$

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The claim: λ is an isomorphism

Thank you very much for your attention!

Se efcharistó pára polý!