Crossed products by $\mathbb{R}_+^*$ and calculus on filtered manifolds

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Athens, Non Commutative Geometry and Representation Theory

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- II - What happens in the noncommutative case ?
- III - Symbols, decomposition and isomorphism

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$$0 \longrightarrow \mathcal{C}_0(\mathbb{R}_+^*, \mathcal{K}) \longrightarrow C^*(\mathbb{T}^+M) \longrightarrow \mathcal{C}_0(T^*M) \longrightarrow 0$$

Zoom action:

$$\alpha_\lambda \colon (x, y, t) \mapsto (x, y, \lambda^{-1} t)$$
$$(x, \xi, 0) \mapsto (x, \lambda\xi, 0)$$

$\alpha_\lambda \in \mathrm{Aut}(\mathbb{T}^+ M), \lambda > 0.$

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$\alpha_\lambda \in \mathrm{Aut}(\mathbb{T}^+M), \lambda > 0$.
Define $C_0^*(\mathbb{T}^+M)$ as the pre-image of $\mathcal{C}_0(T^*M \setminus 0)$. The action (by pullback) restricts to $\mathbb{R}_+^* \curvearrowright C_0^*(\mathbb{T}^+M)$.

Theorem (Debord, Skandalis '13)

There exists an (explicit) Morita equivalence between the crossed product of the exact sequence of the tangent groupoid

$$0 \longrightarrow \mathcal{K} \otimes \mathcal{K} \longrightarrow C_0^*(\mathbb{T}^+ M) \rtimes \mathbb{R}_+^* \longrightarrow \mathcal{C}(\mathbb{S}^* M) \otimes \mathcal{K} \longrightarrow 0$$

and the one of pseudodifferential operators

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Notice the $\mathbb{R}$ action on the algebra of symbols $\mathcal{C}(\mathbb{S}^* M)$ is trivial.

M compact manifold. Filtration by subbundles $H^1 \subset \cdots \subset H^r = TM$ with $\left[\Gamma(H^i), \Gamma(H^j)\right]$. Replace TM by $T_H M \to M$ bundle of nilpotent Lie groups. Analog Connes' tangent groupoid $\mathbb{T}_H M = M \times M \times \mathbb{R}^* \sqcup T_H M \times \{0\} \rightrightarrows M \times \mathbb{R}$ (Choi, Ponge, Van Erp, Yuncken...).

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Family of inhomogeneous dilations $\delta_\lambda = \mathrm{Diag}(\lambda^1, \cdots, \lambda^r)$ replaces scalar multiplication.
$\forall \lambda > 0, \delta_\lambda \in \mathrm{Aut}(T_H M)$.

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Define $C_0^*(\mathbb{T}_H^+M)$ as the pre-image of $C_0^*(T_HM)$ which is the kernel of the trivial representations in $C^*(T_HM)$. The action (by pullback) restricts to $\mathbb{R}_+^* \curvearrowright C_0^*(\mathbb{T}_H^+M)$.

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- Realize elements of $\Sigma(T_H M)$ as multipliers of $C_0^*(T_H M)$
- Define the appropriate $\mathbb{R}$-action
- Show isomorphism

$\pi\colon G \to M$ bundle of nilpotent Lie groups over a compact base.

$\mathfrak{g} = \bigoplus_{i=1}^{r} \mathfrak{g}_i$ with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$.

$\mathcal{D}'(G, \Omega^{1/2}) = \mathcal{C}_c^\infty(G, \Omega^{-1/2} \otimes \Omega^1(TG))'$.

Definition

A symbol of order m is a distribution $\sigma \in \mathcal{D}'(G, \Omega^{1/2})$ which is transversal to π, has proper support and is quasi-homogeneous of degree m w.r.t δ, i.e.

$$\forall \lambda > 0, \delta_{\lambda *}\sigma - \lambda^m \sigma \in \mathcal{C}_c^\infty(G, \Omega^{1/2}).$$

Denote $S^m(G)$ the space of such symbols and $\Sigma^m(G) = {S^m(G)}/{\mathcal{C}_c^\infty(G, \Omega^{1/2})}$ the set of principal symbols.

Convolution of symbols gives composition maps $S^m(G) \times S^\ell(G) \to S^{m+\ell}(G)$.

Symbols act by convolution on $\mathcal{C}^\infty(G, \Omega^{1/2})$.

If $\Re(m) \leq 0$, the operators extent to multipliers of $C^*(G)$ and if $\Re(m) < 0$ they extend to elements of $C^*(G)$. Denote by $S^*(G)$ the C^*-closure of $S^0(G)$ inside $M(C^*(G))$ and $\Sigma(G) = {S^*(G)}/{C^*(G)}$, the later has $\Sigma^0(G)$ as a dense subalgebra.

Lemma (Taylor)

Let $m \in \mathbb{C}$, $u \in S^m(G)$. Then there exists a unique smooth function $v \in \mathcal{C}^\infty(\mathfrak{g}^* \setminus 0)$ such that:

- *v is homogeneous of degree m: $\forall \lambda > 0, v \circ {}^t\delta_\lambda = \lambda^s v$*
- *if $\chi \in \mathcal{C}_c^\infty(\mathfrak{g}^*)$ is equal to 1 on a neighborhood of the zero section then $\hat{u} - (1-\chi)v \in \mathscr{S}(\mathfrak{g}^*)$.*

Corollary

If $m > -n = -\sum_{i \geq 1} i \dim(\mathfrak{g}_i)$ then

$$\Sigma^m(G) \cong \mathcal{K}^m(G) = \{\sigma \in \mathscr{S}'(G, \Omega^{1/2}) \ / \ \forall \lambda > 0, \delta_{\lambda *}\sigma = \lambda^m \sigma\}$$

If $m \leq 0$ convolution by elements of $\mathcal{K}^m(G)$ extend to multipliers of $C_0^*(G)$. The resulting map $\Sigma^0(G) \to M(C_0^*(G))$ extends to $\Sigma(G)$.

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Let $D \in \Sigma^r(G)$ be a positive Rockland differential operator. One can define a symbol $D^{\mathrm{i}t/r} \in \Sigma^{\mathrm{i}t}(G)$ (Dave, Haller). Define $\beta(t) = \mathrm{Ad}(D^{-\mathrm{i}t/r})$ then $\beta(t) \in \mathrm{Aut}(C_0^*(G))$. Taylor's lemma also realizes $D^{\mathrm{i}t/r}$ as a unitary multiplier of $C_0^*(G)$ and we have the duality relation: $\delta_\lambda^* D^{-\mathrm{i}t/r} = \lambda^{\mathrm{i}t} D^{-\mathrm{i}t/r}$.

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We have thus constructed a map $\Sigma(G) \rtimes_\beta \mathbb{R} \to C_0^*(G)$.

Both algebras $\Sigma(G)$ and $C_0^*(G)$ are continuous fields of C^*-algebras over M, the previous map preserves this structure. We can then reduce to the case of a single graded Lie group.

Theorem (Fermanian-Kammerer, Fischer)

Let G be a graded Lie group. For $\pi \in \hat{G} \setminus \{1\}$ denote by $[\pi]$ its class in $(\hat{G} \setminus \{1\})/_{\mathbb{R}_+^*}$. For every $\pi \in \hat{G} \setminus \{1\}$, we have $\pi \circ \varphi \in \widehat{\Sigma(G)}$ where $\varphi\colon \Sigma(G) \to M(C_0^*(G))$ is the map constructed previously. The following map is a homeomorphism:

$$\begin{aligned}(\hat{G} \setminus \{1\})/_{\mathbb{R}_+^*} &\to \widehat{\Sigma(G)} \\ [\pi] &\mapsto \pi \circ \varphi.\end{aligned}$$

Using Kirillov's theory ($\hat{G} \cong \mathfrak{g}^* / G$) and Pedersen's fine stratification of the orbits we obtain the following decomposition of the spectrum:

Theorem (Pedersen)

There exists a filtration $\emptyset = V_0 \subset V_1 \subset \cdots \subset V_d = \hat{G} \setminus \{0\}$ by open subsets such that the each $\Lambda_i = V_i \setminus V_{i-1}$ is Haussdorff, $\mathbb{R}_+^*$-invariant and the action $\mathbb{R}_+^* \curvearrowright \Lambda_i$ is free and proper. In particular the sets $\Lambda_i / \mathbb{R}_+^*$ are Haussdorff.
Moreover (Lipsman, Rosenberg): $\forall i, \exists n \in \mathbb{N}$ s.t. every $\pi \in \Lambda_i$ may be realized on the same Hilbert space $L^2(\mathbb{R}^n)$, smooth vectors are the Schwartz functions $\mathcal{S}(\mathbb{R}^n)$ and elements of $\mathfrak{g}$ are differential operators with polynomial coefficients. These coefficients have a (non-singular) rational dependence when π varies.

Using Pedersen's stratification we obtain decompositions of $C_0^*(G)$ and $\Sigma(G)$:

$$\{0\} = J_0 \triangleleft J_1 \triangleleft \cdots \triangleleft J_d = C_0^*(G)$$
$$\{0\} = \Sigma_0 \triangleleft \Sigma_1 \triangleleft \cdots \triangleleft \Sigma_d = \Sigma(G)$$

with ${}^{J_i}\!/_{J_{i-1}} \cong \mathcal{C}_0(\Lambda_i, \mathcal{K}_i)$ and ${}^{\Sigma_i}\!/_{\Sigma_{i-1}}$ is a continuous field of C^*-algebras over ${}^{\Lambda_i}\!/_{\mathbb{R}_+^*}$.

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β preserves each component, we get maps

$$\left({}^{\Sigma_i}\!/_{\Sigma_{i-1}}\right) \rtimes \mathbb{R} \to {}^{J_i}\!/_{J_{i-1}}.$$

Example: $G = H_3$ the Heisenberg group. Its Lie algebra is generated by X, Y, Z with $[X, Y] = Z$. $\Lambda_1 \cong \mathbb{R}^*, \Lambda_2 \cong \mathbb{R}^2 \setminus \{0\}$.

$$0 \longrightarrow \mathcal{C}_0(\mathbb{R}^*, \mathcal{K}) \to C_0^*(H_3) \longrightarrow \mathcal{C}_0(\mathbb{R}^2 \setminus \{0\}) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{K} \oplus \mathcal{K} \to \Sigma(H_3) \longrightarrow \mathcal{C}_0(\mathbb{S}^1) \longrightarrow 0$$

The second line is the well known Epstein-Melrose decomposition of symbols in the Heisenberg calculus.

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Denote by $A_\lambda = \mathrm{im}(\pi_\lambda)$ the fiber of $\Sigma(G)$ at $[\lambda] \in {}^{\Lambda_i}\!/\!_{\mathbb{R}_+^*}$. We now have to show that $A_\lambda = \mathcal{K}(\mathcal{H}_{\pi_\lambda})$:

- A_λ contains $\mathcal{K}(\mathcal{H}_\lambda)$
- A_λ is a simple algebra

Theorem (Ewert ('20) ; C. ('23))

Let $G \to M$ be a bundle of graded Lie groups then $\Sigma(G) \rtimes \mathbb{R}$ and $C_0^*(G)$ are isomorphic.

Corollary (C. ('23))

Let G be a graded Lie group. Let $\emptyset = V_0 \subset V_1 \subset \cdots \subset V_d = \hat{G} \setminus \{0\}$ be a Pedersen stratification and $\{0\} = \Sigma_0 \triangleleft \Sigma_1 \triangleleft \cdots \triangleleft \Sigma_r = \Sigma(G)$ be the corresponding decomposition of the symbol algebra. Then:

$$\forall i \geq 1, {}^{\Sigma_i}/_{\Sigma_{i-1}} \cong \mathcal{C}({}^{\Lambda_i}/_{\mathbb{R}_+^*}, \mathcal{K}_i).$$

Thank you for your attention !