A Shubin-type calculus on graded Lie groups

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Motivation

Closed manifold		Closed filtered manifold
Hörmander ΨDOs Atiyah-Singer index theorem	$\rightsquigarrow$	calculus by van Erp–Yuncken index theorems by Baum–van Erp, Mohsen, Goffeng-Kuzmin, ...

$\mathbb{R}^n$		G graded Lie group
Shubin ΨDOs index formulas by Fedosov, Hörmander, Elliott–Natsume–Nest	$\rightsquigarrow$	?

Questions:

(1) Can one formulate the Shubin calculus in terms of a tangent groupoid?

(2) Is there a filtered Shubin calculus?

(3) When are operators elliptic in this calculus and how to determine their index?

Shubin calculus

Definition

A function $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is a Shubin symbol of order m if

$$\forall \gamma, \delta \in \mathbb{N}_0^n \, \exists C_{\gamma,\delta} > 0 \colon |\partial_x^\gamma \partial_\xi^\delta a(x,\xi)| \leq C_{\gamma,\delta}(1 + |x| + |\xi|)^{m-|\gamma|-|\delta|}.$$

Example (Differential operators with polynomial coefficients)

$$a(x,\xi) = \sum_{|\alpha|+|\beta|\leq m} c_{\alpha,\beta} x^\alpha (i\xi)^\beta \rightsquigarrow Op(a) = \sum_{|\alpha|+|\beta|\leq m} c_{\alpha,\beta} x^\alpha \partial^\beta$$

- consider classical symbols: homogeneous expansion with respect to the scalings $\lambda \cdot (x,\xi) = (\lambda x, \lambda \xi)$ for $\lambda > 0$,
- principal symbol is a m-homogeneous function on $\mathbb{R}^{2n} \setminus \{0\}$,
- denote $\Psi_S^m = \{\mathrm{Op}(a) \mid a \text{ classical Shubin symbol of order m}\}$,
- the principal symbol map induces short exact sequences

$$0 \to \Psi_S^{m-1} \to \Psi_S^m \overset{\sigma_m}{\to} C^\infty(S^{2n-1}) \to 0.$$

Ellipticity in the Shubin calculus

Definition

A classical Shubin pseudodifferential operator A is elliptic if its principal symbol $\sigma_m(A) \in C^\infty(S^{2n-1})$ is invertible.

- denote by $\mathcal{K}^\infty$ integral operators with kernel in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$,
- $A \in \Psi_S^m$ elliptic $\Rightarrow$ there is a parametrix $B \in \Psi_S^{-m}$ such that $AB - 1 \in \mathcal{K}^\infty$ and $BA - 1 \in \mathcal{K}^\infty$,
- elliptic Shubin operators are Fredholm.

Examples

- on $\mathbb{R}^n$: harmonic oscillator $-\Delta + |x|^2 \in \Psi_S^2$ (principal symbol $|\xi|^2 + |x|^2$),
- on $\mathbb{R}$: creation/annihilation operator $x \pm \partial_x \in \Psi_S^1$ (principal symbol $x \pm i\xi$).

Graded Lie groups

Definition

A graded Lie group is a simply connected Lie group G whose Lie algebra $\mathfrak{g}$ can be written as

$$\mathfrak{g} = \bigoplus_{j=1}^{r} \mathfrak{g}_j \qquad \text{such that } [\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}.$$

- in particular, graded Lie groups are nilpotent,
- in the following identify $X \in \mathfrak{g}$ with the corresponding left-invariant differential operator on G,
- define the order of $X \in \mathfrak{g}_j$ to be j.

Example (Heisenberg group H with Lie algebra $\mathfrak{h}$)

Let $\mathfrak{h}_1$ be generated by X, Y, $\mathfrak{h}_2$ by $Z = [X, Y]$.

Dilations

- the grading induces a dilation action $\delta\colon \mathbb{R}_{>0} \curvearrowright \mathfrak{g}$ defined by
$$\delta_\lambda(X) = \lambda^j X \text{ for } X \in \mathfrak{g}_j,$$
- integrates to $\delta\colon \mathbb{R}_{>0} \curvearrowright G$,
- $X(f \circ \delta_\lambda) = \lambda^j (Xf) \circ \delta_\lambda$ for $X \in \mathfrak{g}_j$ and $f \in C^\infty(G)$.

Definition

A function $f\colon G \setminus \{0\} \to \mathbb{C}$ is m-homogeneous if $f \circ \delta_\lambda = \lambda^m f$ for all $\lambda > 0$.

$\rightsquigarrow$ new notion of order for the polynomials on G.

Example (Heisenberg group)

- X, Y have order 1, Z has order 2,
- the coordinate functions x, y have order 1, z order 2.

Shubin filtration

- Choose a basis $X_1, \dots, X_n$ of $\mathfrak{g}$ s.t.
  - $X_1, \dots, X_{\dim \mathfrak{g}_1}$ is a basis of $\mathfrak{g}_1$,
  - $X_{\dim \mathfrak{g}_1 + 1}, \dots, X_{\dim \mathfrak{g}_1 + \dim \mathfrak{g}_2}$ is a basis of $\mathfrak{g}_2$,
  - $\dots$
- the weights $q_1, \dots, q_n \in \mathbb{N}$ are defined by $X_i \in \mathfrak{g}_{q_i}$,
- identify $\mathbb{R}^n \xrightarrow{\cong} G$ via $(x_1, \dots, x_n) \mapsto \exp\left(\sum_{i=1}^n x_i X_i\right)$,
- for a multi-index $\alpha \in \mathbb{N}_0^n$, write $[\alpha] = \alpha_1 q_1 + \dots + \alpha_n q_n$,
- homogeneous degree of X^α or x^α is $[\alpha]$.

Shubin filtration of differential operators with polynomial coefficients on G

For $m \in \mathbb{N}_0$, let

$$\mathcal{A}_m = \left\{ \sum_{[\alpha]+[\beta] \leq m} c_{\alpha,\beta} x^\alpha X^\beta \mid c_{\alpha,\beta} \in \mathbb{C} \right\} \subseteq \mathcal{A}_{m+1} \subseteq \dots$$

Rockland condition

Definition

Let P be a left-invariant differential operator on a graded Lie group G. Then P satisfies the Rockland condition if for every $\pi \in \widehat{G} \setminus \{\pi_{\mathrm{triv}}\}$ the operator $\mathrm{d}\pi(P)$ is injective on $\mathcal{H}_\pi^\infty$.

Theorem (Helffer–Nourrigat, Christ–Geller–Głowacki–Polin, Dave–Haller,…)

Let M be a filtered manifold and $P \in \Psi_H^m(M)$. If all model operators $\sigma(P)_x$ satisfy the Rockland condition on the osculating groups G_x, then P is hypoelliptic.

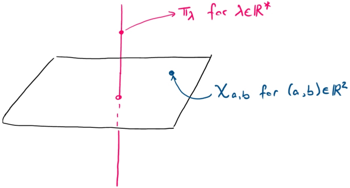
Shubin filtration: obtain an isomorphism

$$\bigoplus_{m=0}^{\infty} \mathcal{A}_m/\mathcal{A}_{m-1} \to \mathcal{U}(\mathfrak{g}^*) \otimes \mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g}^* \oplus \mathfrak{g})$$

induced by $\sum_{[\alpha]+[\beta]\leq m} c_{\alpha\beta} x^\alpha X^\beta \mapsto \sum_{[\alpha]+[\beta]=m} c_{\alpha\beta} (-i\partial_x)^\alpha X^\beta$

$\rightsquigarrow$ Rockland condition on $\mathbb{R}^n \times G$

Example: Heisenberg group



Representations of H

$\widehat{H}$ consists of

- characters $\chi_{a,b}$ on $\mathcal{H} = \mathbb{C}$ for $(a, b) \in \mathbb{R}^2$:
  $X \mapsto ia$, $Y \mapsto ib$, $Z \mapsto 0$,
- Schrödinger representations π_λ on $\mathcal{H} = L^2(\mathbb{R})$ for $\lambda \in \mathbb{R}^*$:
  $X \mapsto \sqrt{|\lambda|}\partial_u$, $Y \mapsto \pm i\sqrt{|\lambda|}u$, $Z \mapsto i\lambda 1$.

$\rightsquigarrow$ can check which operators of the form $-X^2 - Y^2 + \alpha Z + p(x, y, z)$ for $\alpha \in \mathbb{C}$ and a polynomial potential p satisfy the Rockland condition.

Shubin tangent groupoid and zoom action

Consider first $G = \mathbb{R}^n$.

Hörmander classes

- homogeneity of the symbols wrt $\lambda \cdot (x, \xi) = (x, \lambda\xi)$,
- tangent groupoid $\mathbb{TR}^n = (\mathbb{R}^n \times \mathbb{R}) \rtimes \mathbb{R}^n$ for $(x, t) \cdot v = (x + tv, t)$,
- at $t = 0$: $T\mathbb{R}^n$,
- for $t \neq 0$: isomorphic to the pair groupoid $\mathbb{R}^n \times \mathbb{R}^n$ via $(x, t, v) \mapsto (x, x + tv)$,
- zoom action of $\mathbb{R}_{>0}$ $\lambda \cdot (x, t, v) = (x, \frac{t}{\lambda}, \lambda v)$.

Shubin classes

- homogeneity of the symbols wrt $\lambda \cdot (x, \xi) = (\lambda x, \lambda\xi)$,
- Shubin tangent groupoid $\mathbb{T}_S\mathbb{R}^n = (\mathbb{R}^n \times \mathbb{R}) \rtimes \mathbb{R}^n$ for $(x, t) \cdot v = (x + t^2 v, t)$,
- at $t = 0$: $T\mathbb{R}^n$,
- for $t \neq 0$: isomorphic to the pair groupoid $\mathbb{R}^n \times \mathbb{R}^n$ via $(x, t, v) \mapsto (x, x + t^2 v)$,
- Shubin zoom action of $\mathbb{R}_{>0}$ $\lambda \cdot (x, t, v) = (\lambda^{-1} x, \frac{t}{\lambda}, \lambda v)$.

Shubin tangent groupoid and zoom action

Similary, we define for a graded Lie group G using the dilations δ:

- a Shubin tangent groupoid $\mathbb{T}_S G = (G \times \mathbb{R}) \rtimes G$ for $(x, t) \cdot v = (x \cdot \delta_{t^2}(v), t)$,
- a Shubin zoom action $\alpha_\lambda(x, t, v) = (\delta_{\lambda^{-1}}(x), \frac{t}{\lambda}, \delta_\lambda(v))$.

Remark

More generally, we can consider two commuting dilations, one to define the order of left-invariant differential operators, the other for the order of polynomials.

Example

$\mathbb{R}^n$ with different weights $\rightsquigarrow$ anisotropic calculus (Boggiatto–Nicola)

Shubin pseudo-differential calculus

Given this tangent groupoid and zoom action, one can follow the approach of Debord–Skandalis and van Erp–Yuncken to define a corresponding pseudodifferential calculus.

Definition

An operator $P\colon \mathcal{S}(G) \to \mathcal{S}(G)$ is a Shubin type operator of order m ($P \in \Psi^m_S(G)$) if there is a $\mathbb{P} \in \mathcal{K}(\mathbb{T}_S G)$ such that $\mathrm{Op}(\mathbb{P}_1) = P$ and $\alpha_{\lambda *}(\mathbb{P}) - \lambda^m \mathbb{P} \in \mathcal{S}(\mathbb{T}_S G)$ for all $\lambda > 0$.

Here, $\mathcal{K}(\mathbb{T}_S G)$ denotes a certain space of fibred distributions.

Example (Differential operator with polynomial coefficients)

$P = \sum_{[\alpha]+[\beta]\leq m} c_{\alpha,\beta} x^\alpha X^\beta \in \Psi^m_S(G)$: it extends to the zoom-homogeneous $\mathbb{P}$ with $\mathbb{P}_t = \sum_{[\alpha]+[\beta]\leq m} t^{m-[\alpha]-[\beta]} c_{\alpha,\beta} x^\alpha X^\beta_v$.

Properties of the calculus

Analogously to the results of van Erp–Yuncken for filtered manifolds one can show

- there is a well-defined principal cosymbol:
$$\sigma_m(P) = [\mathbb{P}_0] \in \mathcal{K}_0(T_H G)/\mathcal{S}(T_H G)$$
for any essentially homogeneous extension $\mathbb{P}$,
- the principal symbol map induces short exact sequences,
- call P Shubin H-elliptic if $\sigma_m(P)$ is invertible,
- $\Psi^k_S(G)\Psi^l_S(G) \subseteq \Psi^{k+l}_S(G)$,
- $\cap_{m\in\mathbb{Z}}\Psi^m_S(G) = \mathcal{K}^\infty$.

Remark

The C^*-completion of the order zero extension can also be obtained by a generalized fixed point algebra approach. In particular, operators of order 0 are bounded on $L^2(G)$ and of negative order are compact.

Ellipticity and Rockland condition

Using the result of Christ-Geller-Głowacki-Polin:

Theorem

$P \in \Psi_S^m(G)$ is Shubin H-elliptic if and only if $(x,\pi)(\sigma_m(P))$ and $(x,\pi)(\sigma_m(P^t))$ are injective on $\mathcal{H}_\pi^\infty$ for all $(x,\pi) \in G \times \widehat{G} \setminus \{(0,\pi_{\mathrm{triv}})\}$ (Rockland condition).

Example (Analogue of harmonic oscillator)

Fix a common multiple q of the weights $q_1,\dots,q_n \in \mathbb{N}$. Then

$$P = \sum_{j=1}^{n} (-1)^{\frac{q}{q_j}} X_j^{\frac{2q}{q_j}} + \sum_{j=1}^{n} x_j^{\frac{2q}{q_j}} = \mathcal{R} + \|x\|^{2q} \in \Psi_S^{2q}(G)$$

satisfies the Rockland condition.

One can define a Sobolev scale H^s for the Shubin calculus following Dave–Haller for filtered manifolds.

Theorem

Let $P \in \Psi_S^m(G)$ be elliptic. Then $P\colon H^s \to H^{s-m}$ is Fredholm for $s \in \mathbb{R}$.

How to compute the index?

For the Shubin calculus on $\mathbb{R}^n$:

Theorem (Elliott–Natsume–Nest)

Let $\mathrm{Op}(a) \in \Psi_S^m(\mathbb{R}^n)$ *be elliptic of positive order. Then*

$$\mathrm{ind}(\mathrm{Op}(a)) = \frac{1}{(2\pi i)^n n!} \int_{T^*\mathbb{R}^n} \mathrm{tr}(p_a (dp_a)^{2n}),$$

where

$$p_a = \begin{pmatrix} (1+a^*a)^{-1} & (1+a^*a)^{-1}a^* \\ a(1+a^*a)^{-1} & a(1+a^*a)^{-1}a^* \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Idea of the proof for $G = \mathbb{R}^n$

- connect $\mathrm{Op}(a)$ to its symbol a through the tangent groupoid. Let $a_t(x,\xi) = a(x, t^2\xi)$,
- denote for an elliptic operator T
$$p_T = \begin{pmatrix} (1+T^*T)^{-1} & (1+T^*T)^{-1}T^* \\ T(1+T^*T)^{-1} & T(1+T^*T)^{-1}T^* \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$
- then the following defines a continuous section of $C^*(\mathbb{T}_S\mathbb{R}^n)$
$$t \mapsto \begin{cases} p_{\mathrm{Op}(a_t)} & t \neq 0 \\ p_a & t = 0, \end{cases}$$
- use a cyclic cocycle ω such that $\mathrm{ind}(\mathrm{Op}(a)) = \langle p_{\mathrm{Op}(a)}, \omega\rangle$ and $\lim_{t\to 0} \langle p_{\mathrm{Op}(a_t)}, \omega\rangle$ exists.

The cyclic $2n$-cocycle for $G = \mathbb{R}^n$

- for $j = 1, \ldots, n$ define derivations of $\mathcal{K}^\infty$ by $\delta_{2j-1}(T) = [\partial_{x_j}, T]$ and $\delta_{2j}(T) = [x_j, T]$,
- $\omega(T_0, \ldots, T_{2n}) = \dfrac{(-1)^n}{n!} \sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) \operatorname{Tr}(T_0 \delta_{\sigma(1)}(T_1) \ldots \delta_{\sigma(2n)}(T_{2n}))$,
- denote by $\rho_t(f)$ for $t \neq 0$ and $f \in \mathcal{S}(\mathbb{T}_S \mathbb{R}^n)$ their representations as operators on $L^2(\mathbb{R}^n)$

$$\rho_t(f)\psi(x) = (2\pi)^{-n} t^{-2n} \int f(x, t, \tfrac{y-x}{t^2}) \psi(y) \, dy,$$

- then $[\partial_{x_j}, \rho_t(f)] = \rho_t(\partial_{x_j} f)$ and $[x_j, \rho_t(f)] = \rho_t(-t^2 v_j f)$.

The cyclic $2n$-cocycle for $G = \mathbb{R}^n$

For $f_0, \ldots, f_{2n} \in \mathcal{S}(G \times G)$

$$
\begin{aligned}
&\operatorname{Tr}(\rho_t(f_0)\delta_1(\rho_t(f_1)) \ldots \delta_{2n}(\rho_t(f_{2n}))) \\
&= (-t^2)^n \operatorname{Tr}(\rho_t(f_0 *_t \partial_{x_1} f_1 *_t v_1 f_2 *_t \ldots *_t v_n f_{2n})) \\
&= (-2\pi)^n \int_{\mathbb{R}^n} (f_0 *_t \partial_{x_1} f_1 *_t v_1 f_2 *_t \ldots *_t v_n f_{2n})(x,0)\, \mathrm{d}x \\
&\underset{t \to 0}{\to} \frac{(-1)^n}{(2\pi i)^n} \int_{T^*\mathbb{R}^n} \widehat{f_0} \partial_{x_1} \widehat{f_1} \partial_{\xi_1} \widehat{f_2} \ldots \partial_{\xi_n} \widehat{f_{2n}}\, \mathrm{d}x\, \mathrm{d}\xi
\end{aligned}
$$

Consequently, the limit of $\omega(\rho_t(f_0), \ldots, \rho_t(f_{2n}))$ as $t \to 0$ gives the result.

Constructing a cocycle for a graded Lie group G

As G is graded, $G \cong ((\mathbb{R} \rtimes \mathbb{R}) \rtimes \ldots) \rtimes \mathbb{R}$, similarly also $\mathbb{T}_S G$ and their corresponding Schwartz convolution algebras.

Theorem (Elliott–Natsume–Nest)

If $\mathcal{A}$ is a Fréchet algebra with a smooth $\mathbb{R}$-action, then there is an (explicitly constructed) isomorphism $\#\colon HC^k_{\text{per}}(\mathcal{A}) \to HC^{k+1}_{\text{per}}(\mathcal{A} \rtimes \mathbb{R})$. It is compatible with the Connes–Thom isomorphism in K-theory.

We use this to get a cyclic $2n$-cocycle ω_t on $\mathcal{S}(G) \rtimes_t G$ for all t such that

- the pairing with the graph projection at $t = 1$ gives the index,
- it recovers the considered cocycle for $G = \mathbb{R}^n$.

Theorem (E–Nest–Schmitt)

Let $P \in \Psi^m_S(G)$ be of positive order and Shubin-H-elliptic. Then

$$\operatorname{ind}(P) = (\omega_0 \# \operatorname{tr})(p_{\mathbb{P}_0}, \ldots, p_{\mathbb{P}_0}).$$

Example (Heisenberg group)

One computes for $f_i \in \mathcal{S}(G \times G)$

$$\omega_0(f_0, \ldots, f_6) = \sum_{\sigma \in S_6} \operatorname{sgn}(\sigma) \int_G f_0 * D_{\sigma(1)} f_1 * \ldots * D_{\sigma(6)} f_6(x, 0)\, \mathrm{d}x$$
$$+ \text{extra terms}$$

where $D_1 = \partial_{x_1}, D_2 = v_1, D_3 = \partial_{x_2}, D_4 = v_2, D_5 = \partial_{x_3}, D_6 = v_3$.

Heisenberg group

$$\omega_0(f_0,\dots,f_6)=\sum_i c_i\int_{\mathbb{R}^n}(f_0*D_1^if_1*\cdots*D_6^if_6)(x,0)\,\mathrm{d}x,$$

where $c_i=(-1)^{a+b+c}\operatorname{sgn}(\sigma)d_i$ with a,b,c the positions of the elements in the top row, σ the sign of the permutation in the top row, and

f_1	f_2	f_3	f_4	f_5	f_6	d_i
∂_{x_2}		∂_{x_1}	∂_{x_3}			sign of permutation in bottom row
	v_3			v_1	v_2	
	∂_{x_1}		∂_{x_3}	∂_{x_2}		$(-1)^{k+\ell+1}/2$; k,ℓ indices of left, right element in bottom row
v_2		v_1v_1			v_2	
	∂_{x_2}	∂_{x_3}			∂_{x_1}	$(-1)^{k+\ell}/2$; k,ℓ index of v_1 in bottom row $(k,\ell)\neq(1,2),(3,4)$
v_1	v_2		v_2	v_1		

2nd or 3rd element contained in column with 2 entries

Rewriting the cocycle using Fourier transform

Plancherel Theorem

For $f \in \mathcal{S}(G)$ and the *Plancherel measure* on $\widehat{G}$

$$f(0) = \int_{\widehat{G}} \operatorname{Tr}\left(\widehat{f}(\pi)\right) \mathrm{d}\mu(\pi).$$

- denote by $\Delta_{v_i}\widehat{f}(\pi) = \widehat{v_i \cdot f}(\pi)$ (difference operators),
- the Plancherel measure on $\widehat{H}$ is supported within the Schrödinger representations π_λ for $\lambda \in \mathbb{R} \setminus \{0\}$,
- using this, the cocycle can be rewritten, for example,

$$\int_G f_0 * \partial_{x_1} f_1 * \ldots * v_3 f_6(x,0)\,\mathrm{d}x$$
$$= (2\pi)^{-4} \int_{H \times \mathbb{R}\setminus\{0\}} \operatorname{sgn}(\lambda) \operatorname{Tr}(\widehat{f_0(x)}(\pi_\lambda)\partial_{x_1}\widehat{f_1(x)}(\pi_\lambda) \ldots \Delta_{v_3}\widehat{f_6(x)}(\pi_\lambda))\,\mathrm{d}x\,\mathrm{d}\lambda.$$

Thank you!

References

[DH22] Shantanu Dave and Stefan Haller, *Graded hypoellipticity of BGG sequences*, Annals of Global Analysis and Geometry **62** (2022), no. 4, 721–789.

[DS14] Claire Debord and Georges Skandalis, *Adiabatic groupoid, crossed product by $\mathbb{R}_+^*$ and pseudodifferential calculus*, Adv. Math. **257** (2014), 66–91, DOI 10.1016/j.aim.2014.02.012. MR3187645

[ENN88] G.A. Elliott, T. Natsume, and R. Nest, *Cyclic cohomology for one-parameter smooth crossed products*, Acta Mathematica **160** (1988), no. 1, 285–305.

[ENN96] ______, *The Atiyah-Singer index theorem as passage to the classical limit in quantum mechanics*, Communications in mathematical physics **182** (1996), 505–533.

[vEY19] Erik van Erp and Robert Yuncken, *A groupoid approach to pseudodifferential calculi*, J. Reine Angew. Math. **756** (2019), 151–182, DOI 10.1515/crelle-2017-0035. MR4026451

[HN79] Bernard Helffer and Jean F. Nourrigat, *Caracterisation des opérateurs hypoelliptiques homogènes invariants à gauche sur un groupe de Lie nilpotent gradué*, Comm. Partial Differential Equations **4** (1979), no. 8, 899–958, DOI 10.1080/03605307908820115 (French). MR537467

[Moh] Omar Mohsen, *Index theorem for inhomogeneous hypoelliptic differential operators*, Münster Journal of Mathematics.

[Shu01] Mikhail A. Shubin, *Pseudodifferential operators and spectral theory*, 2nd ed., Springer-Verlag, Berlin, 2001. Translated from the 1978 Russian original by Stig I. Andersson. MR1852334