The Helffer-Nourrigat conjecture: hypoellipticity of polynomials of vector fields

Robert Yuncken

Université de Lorraine, Metz

Athens, March 2023

Hypoelliptic differential operators

Hypoelliptic differential operators

M — smooth manifold. Recall...

Definition (Hypoellipticity)

A differential operator P on M is **hypoelliptic** if the differential equation

$$Pu = f$$

has the smoothness of solutions property :

$$f|_V \in C^\infty \Rightarrow u|_V \in C^\infty$$

for any open subset $V \subseteq M$.

Hypoelliptic differential operators

M — smooth manifold. Recall...

Definition (Hypoellipticity)

A differential operator P on M is **hypoelliptic** if the differential equation

$$Pu = f$$

has the smoothness of solutions property :

$$f|_V \in C^\infty \Rightarrow u|_V \in C^\infty$$

for any open subset $V \subseteq M$.

Remark.
P is **maximally hypoelliptic** if moreover we have Sobolev estimates.

Examples on $\mathbb{R}^2$

The Laplace operator $P = -\partial_x^2 - \partial_y^2$ is hypoelliptic.

Examples on $\mathbb{R}^2$

The Laplace operator $P = -\partial_x^2 - \partial_y^2$ is hypoelliptic.

The heat operator $P = \partial_t - \partial_x^2$ is hypoelliptic.

Examples on $\mathbb{R}^2$

The Laplace operator $P = -\partial_x^2 - \partial_y^2$ is hypoelliptic.

The heat operator $P = \partial_t - \partial_x^2$ is hypoelliptic.

The wave operator $P = \partial_t^2 - \partial_x^2$ is **not** hypoelliptic.

Examples on $\mathbb{R}^2$

The Laplace operator $P = -\partial_x^2 - \partial_y^2$ is hypoelliptic.

The heat operator $P = \partial_t - \partial_x^2$ is hypoelliptic.

The wave operator $P = \partial_t^2 - \partial_x^2$ is **not** hypoelliptic.

A hypoelliptic Laplacian $P = -\partial_x^2 - x\partial_y$ is hypoelliptic.

Examples on $\mathbb{R}^2$

The Laplace operator	$P = -\partial_x^2 - \partial_y^2$ is hypoelliptic.
The heat operator	$P = \partial_t - \partial_x^2$ is hypoelliptic.
The wave operator	$P = \partial_t^2 - \partial_x^2$ is **not** hypoelliptic.
A hypoelliptic Laplacian	$P = -\partial_x^2 - x\partial_y$ is hypoelliptic.
The Grushin sublaplacian	$P = -\partial_x^2 - x^2\partial_y^2 + \alpha\partial_y$ is hypoelliptic iff $\alpha \notin \{\pm i, \pm 3i, \pm 5i, \ldots\}$.

Examples on $\mathbb{R}^2$

The Laplace operator	$P = -\partial_x^2 - \partial_y^2$ is hypoelliptic.
The heat operator	$P = \partial_t - \partial_x^2$ is hypoelliptic.
The wave operator	$P = \partial_t^2 - \partial_x^2$ is **not** hypoelliptic.
A hypoelliptic Laplacian	$P = -\partial_x^2 - x\partial_y$ is hypoelliptic.
The Grushin sublaplacian	$P = -\partial_x^2 - x^2\partial_y^2 + \alpha\partial_y$ is hypoelliptic iff $\alpha \notin \{\pm i, \pm 3i, \pm 5i, \ldots\}$.

Q. What governs hypoellipticity?

A. [Rockland, **Helffer-Nourrigat**]: Representation theory of osculating nilpotent Lie groups.

The Helffer-Nourrigat Conjecture

Given vector fields $V_1, \ldots, V_n \in \Gamma^\infty(M)$, Helffer-Nourrigat construct:

The Helffer-Nourrigat Conjecture

Given vector fields $V_1, \ldots, V_n \in \Gamma^\infty(M)$, Helffer-Nourrigat construct:

- an "osculating" nilpotent Lie group $\mathrm{Gr}\mathcal{F}_p$ at each $p \in M$ (depending on the infinitesimal geometry of the vector fields V_i),

The Helffer-Nourrigat Conjecture

Given vector fields $V_1, \ldots, V_n \in \Gamma^\infty(M)$, Helffer-Nourrigat construct:

- an "osculating" nilpotent Lie group $\mathsf{Gr}\mathcal{F}_p$ at each $p \in M$ (depending on the infinitesimal geometry of the vector fields V_i),
- a family $\mathrm{HN}(\mathcal{F})_p \subseteq \widehat{\mathsf{Gr}\mathcal{F}_p}$ of unitary irreps of each osculating group (depending on the singular geometry of the vector fields V_i),

The Helffer-Nourrigat Conjecture

Given vector fields $V_1, \ldots, V_n \in \Gamma^\infty(M)$, Helffer-Nourrigat construct:

- an "osculating" nilpotent Lie group $\mathrm{Gr}\mathcal{F}_p$ at each $p \in M$ (depending on the infinitesimal geometry of the vector fields V_i),
- a family $\mathrm{HN}(\mathcal{F})_p \subseteq \widehat{\mathrm{Gr}\mathcal{F}_p}$ of unitary irreps of each osculating group (depending on the singular geometry of the vector fields V_i),

such that:

The Helffer-Nourrigat conjecture

A polynomial P in V_i is maximally hypoelliptic iff $\pi([P]_p)$ is left-invertible on $\mathcal{H}_\pi^\infty$ for every $\pi \in \bigsqcup_p \mathrm{HN}(\mathcal{F})_p$.

The Helffer-Nourrigat Conjecture

Given vector fields $V_1, \ldots, V_n \in \Gamma^\infty(M)$, Helffer-Nourrigat construct:

- an "osculating" nilpotent Lie group $\mathrm{Gr}\mathcal{F}_p$ at each $p \in M$ (depending on the infinitesimal geometry of the vector fields V_i),
- a family $\mathrm{HN}(\mathcal{F})_p \subseteq \widehat{\mathrm{Gr}\mathcal{F}_p}$ of unitary irreps of each osculating group (depending on the singular geometry of the vector fields V_i),

such that:

The Helffer-Nourrigat conjecture

A polynomial P in V_i is maximally hypoelliptic iff $\pi([P]_p)$ is left-invertible on $\mathcal{H}_\pi^\infty$ for every $\pi \in \bigsqcup_p \mathrm{HN}(\mathcal{F})_p$.

Remarks.

1. Here, $[P]_p$ is **an** image of P in the enveloping algebra $\mathcal{U}(\mathfrak{gr}\mathcal{F}_p)$.

The Helffer-Nourrigat Conjecture

Given vector fields $V_1, \dots, V_n \in \Gamma^\infty(M)$, Helffer-Nourrigat construct:

- an "osculating" nilpotent Lie group $\mathrm{Gr}\mathcal{F}_p$ at each $p \in M$ (depending on the infinitesimal geometry of the vector fields V_i),
- a family $\mathrm{HN}(\mathcal{F})_p \subseteq \widehat{\mathrm{Gr}\mathcal{F}_p}$ of unitary irreps of each osculating group (depending on the singular geometry of the vector fields V_i),

such that:

The Helffer-Nourrigat conjecture

A polynomial P in V_i is maximally hypoelliptic iff $\pi([P]_p)$ is left-invertible on $\mathcal{H}_\pi^\infty$ for every $\pi \in \bigsqcup_p \mathrm{HN}(\mathcal{F})_p$.

Remarks.

1. Here, $[P]_p$ is **an** image of P in the enveloping algebra $\mathcal{U}(\mathfrak{gr}\mathcal{F}_p)$.
2. Helffer-Nourrigat's construction of $\mathrm{HN}(\mathcal{F})_p$ is essentially a *DNC*.

The Helffer-Nourrigat Conjecture

Given vector fields $V_1, \ldots, V_n \in \Gamma^\infty(M)$, Helffer-Nourrigat construct:

- an "osculating" nilpotent Lie group $\mathrm{Gr}\mathcal{F}_p$ at each $p \in M$ (depending on the infinitesimal geometry of the vector fields V_i),
- a family $\mathrm{HN}(\mathcal{F})_p \subseteq \widehat{\mathrm{Gr}\mathcal{F}_p}$ of unitary irreps of each osculating group (depending on the singular geometry of the vector fields V_i),

such that:

The Helffer-Nourrigat conjecture

A polynomial P in V_i is maximally hypoelliptic iff $\pi([P]_p)$ is left-invertible on $\mathcal{H}_\pi^\infty$ for every $\pi \in \bigsqcup_p \mathrm{HN}(\mathcal{F})_p$.

Remarks.

1. Here, $[P]_p$ is **an** image of P in the enveloping algebra $\mathcal{U}(\mathfrak{gr}\mathcal{F}_p)$.
2. Helffer-Nourrigat's construction of $\mathrm{HN}(\mathcal{F})_p$ is essentially a *DNC*.
3. Thus, we can prove the conjecture using the groupoid approach to ΨD calculus (pioneered by Connes and Debord-Skandalis).

The groupoid approach to pseudodifferential operators

What is a pseudodifferential calculus?

A **pseudodifferential calculus** on M is an extension of the $\mathbb{N}$-filtered algebra $\mathrm{DO}^m(M)$ to a $\mathbb{Z}$-filtered algebra $\Psi^m(M)$ s.t.

What is a pseudodifferential calculus?

A **pseudodifferential calculus** on M is an extension of the $\mathbb{N}$-filtered algebra $\mathrm{DO}^m(M)$ to a $\mathbb{Z}$-filtered algebra $\Psi^m(M)$ s.t.

Regularity: $\Psi^{-\infty} := \bigcap_{m\in\mathbb{Z}} \Psi^m(M) = \{\text{smoothing ops}\}$.

What is a pseudodifferential calculus?

A **pseudodifferential calculus** on M is an extension of the $\mathbb{N}$-filtered algebra $\mathrm{DO}^m(M)$ to a $\mathbb{Z}$-filtered algebra $\Psi^m(M)$ s.t.

Regularity: $\Psi^{-\infty} := \bigcap_{m\in\mathbb{Z}} \Psi^m(M) = \{\text{smoothing ops}\}$.

Pseudolocality: Schwartz kernels of $P \in \Psi^m(M)$ are smooth off the diag.

What is a pseudodifferential calculus?

A **pseudodifferential calculus** on M is an extension of the $\mathbb{N}$-filtered algebra $\mathrm{DO}^m(M)$ to a $\mathbb{Z}$-filtered algebra $\Psi^m(M)$ s.t.

Regularity: $\Psi^{-\infty} := \bigcap_{m\in\mathbb{Z}} \Psi^m(M) = \{\text{smoothing ops}\}$.

Pseudolocality: Schwartz kernels of $P \in \Psi^m(M)$ are smooth off the diag.

Parametrices: $P \in \mathrm{DO}^m(M)$ with invertible principal symbol $\Rightarrow \exists Q \in \Psi^{-m}(M)$ such that $QP - I \in \Psi^{-\infty}(M)$.

What is a pseudodifferential calculus?

A **pseudodifferential calculus** on M is an extension of the $\mathbb{N}$-filtered algebra $\mathrm{DO}^m(M)$ to a $\mathbb{Z}$-filtered algebra $\Psi^m(M)$ s.t.

Regularity: $\Psi^{-\infty} := \bigcap_{m\in\mathbb{Z}} \Psi^m(M) = \{\text{smoothing ops}\}$.

Pseudolocality: Schwartz kernels of $P \in \Psi^m(M)$ are smooth off the diag.

Parametrices: $P \in \mathrm{DO}^m(M)$ with invertible principal symbol $\Rightarrow \exists Q \in \Psi^{-m}(M)$ such that $QP - I \in \Psi^{-\infty}(M)$.

Why?

What is a pseudodifferential calculus?

A **pseudodifferential calculus** on M is an extension of the $\mathbb{N}$-filtered algebra $\mathrm{DO}^m(M)$ to a $\mathbb{Z}$-filtered algebra $\Psi^m(M)$ s.t.

Regularity: $\Psi^{-\infty} := \bigcap_{m\in\mathbb{Z}} \Psi^m(M) = \{\text{smoothing ops}\}$.

Pseudolocality: Schwartz kernels of $P \in \Psi^m(M)$ are smooth off the diag.

Parametrices: $P \in \mathrm{DO}^m(M)$ with invertible principal symbol
$\Rightarrow \exists Q \in \Psi^{-m}(M)$ such that $QP - I \in \Psi^{-\infty}(M)$.

Why? Given a pseudodifferential calculus, we can prove:

Theorem

Invertible principal symbol ("elliptic") $\Rightarrow$ Hypoelliptic.

What is a pseudodifferential calculus?

A **pseudodifferential calculus** on M is an extension of the $\mathbb{N}$-filtered algebra $\mathrm{DO}^m(M)$ to a $\mathbb{Z}$-filtered algebra $\Psi^m(M)$ s.t.

Regularity: $\Psi^{-\infty} := \bigcap_{m\in\mathbb{Z}} \Psi^m(M) = \{\text{smoothing ops}\}$.

Pseudolocality: Schwartz kernels of $P \in \Psi^m(M)$ are smooth off the diag.

Parametrices: $P \in \mathrm{DO}^m(M)$ with invertible principal symbol $\Rightarrow \exists Q \in \Psi^{-m}(M)$ such that $QP - I \in \Psi^{-\infty}(M)$.

Why? Given a pseudodifferential calculus, we can prove:

Theorem

Invertible principal symbol ("elliptic") ⇒ Hypoelliptic.

Proof.

Given $Pu = f$, write $u = \underbrace{(I - QP)u}_{\substack{\text{smooth}\\ \text{by regularity}}} + \underbrace{Q(Pu)}_{\substack{\text{same sing. supp. as } f = Pu\\ \text{by pseudolocality}}}$ □

Groupoids and ΨDOs

Two revolutionary ideas. . .

Groupoids and ΨDOs

Two revolutionary ideas...

Connes (1980s)

A pseudodifferential operator P and its principal symbol $\sigma^m(P)$ are the restrictions at $t = 1$ and $t = 0$, respectively, of a larger deformation family $(\mathbb{P}_t)$ over $t \in \mathbb{R}$.

$\rightsquigarrow$ Connes' tangent groupoid $\mathbb{T}M$.

Groupoids and ΨDOs

Two revolutionary ideas...

Connes (1980s)

A pseudodifferential operator P and its principal symbol $\sigma^m(P)$ are the restrictions at $t = 1$ and $t = 0$, respectively, of a larger deformation family $(\mathbb{P}_t)$ over $t \in \mathbb{R}$.

$\rightsquigarrow$ Connes' tangent groupoid $\mathbb{T}M$.

Debord-Skandalis (2010s)

Can characterize these deformation families $(\mathbb{P}_t)$ using a canonical $\mathbb{R}_+^\times$-action on $\mathbb{T}M$.

A basic fact about homogeneous polynomials

Proposition

There is a 1–1 correspondence

$$\left\{ \begin{array}{c} \textit{polynomials on } \mathbb{R}^n \textit{ of} \\ \textit{order} \leqslant m \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{homog. polynomials on } \mathbb{R}^{n+1} \\ \textit{of degree} = m \end{array} \right\}$$

$$a(\xi_1, \ldots, \xi_n) = \underline{a}(\xi_1, \ldots, \xi_n, 1) \longleftarrow \underline{a}(\xi_1, \ldots, \xi_n, t)$$

given by restriction at $t = 1$.

A basic fact about homogeneous polynomials

Proposition

There is a 1–1 correspondence

$$\left\{ \begin{array}{c} \textit{polynomials on } \mathbb{R}^n \textit{ of} \\ \textit{order} \leqslant m \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{homog. polynomials on } \mathbb{R}^{n+1} \\ \textit{of degree} = m \end{array} \right\}$$

$$a(\xi_1, \ldots, \xi_n) = \underline{a}(\xi_1, \ldots, \xi_n, 1) \quad \longleftarrow \quad \underline{a}(\xi_1, \ldots, \xi_n, t)$$

given by restriction at $t = 1$.

Remark.

1. The inverse is given by $x^\alpha \mapsto t^{m-|\alpha|}x^\alpha$ for any multi-index $\alpha \in \mathbb{N}^n$.

A basic fact about homogeneous polynomials

Proposition

There is a 1–1 correspondence

$$\left\{ \begin{array}{c} \textit{polynomials on } \mathbb{R}^n \textit{ of} \\ \textit{order} \leqslant m \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{homog. polynomials on } \mathbb{R}^{n+1} \\ \textit{of degree} = m \end{array} \right\}$$

$$a(\xi_1, \ldots, \xi_n) = \underline{a}(\xi_1, \ldots, \xi_n, 1) \quad \longleftarrow \quad \underline{a}(\xi_1, \ldots, \xi_n, t)$$

given by restriction at $t = 1$.

Remark.

1. The inverse is given by $x^\alpha \mapsto t^{m-|\alpha|}x^\alpha$ for any multi-index $\alpha \in \mathbb{N}^n$.
2. The restriction $\underline{a}(\xi_1, \ldots, \xi_n, 0)$ at $t = 0$ is the **principal part** of a.

A basic fact about polyhomogeneous symbols

Recall: A classical ΨDO on $\mathbb{R}^n$ is an operator of the form

$$Pf(x) = \int_{\mathbb{R}^n} a(x,\xi)\widehat{f}(\xi)e^{i\langle \xi,x\rangle}\,d\xi,$$

where $a(x,\xi)$ is a **polyhomog. symbol** (generalizing a polynomial in ξ).

A basic fact about polyhomogeneous symbols

Recall: A classical ΨDO on $\mathbb{R}^n$ is an operator of the form

$$Pf(x) = \int_{\mathbb{R}^n} a(x,\xi)\hat{f}(\xi)e^{i\langle \xi, x\rangle}\, d\xi,$$

where $a(x,\xi)$ is a **polyhomog. symbol** (generalizing a polynomial in ξ).

Theorem (Couchet-Y., following Van Erp-Y.)

There is a 1–1 correspondence

$$\left\{ \begin{array}{c} \textit{polyhomog. symbols on} \\ \mathbb{R}^n \times \mathbb{R}^n \textit{ of order} \leqslant m \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{smooth functions on} \\ \mathbb{R}^n \times \mathbb{R}^{n+1} \textit{ homog. of} \\ \textit{degree} = m \textit{ mod Schwartz} \end{array} \right\}$$

$$a(x,\xi) = \underline{a}(x,\xi,1) \quad \longleftarrow \quad \underline{a}(x,\xi,t)$$

A basic fact about polyhomogeneous symbols

Recall: A classical ΨDO on $\mathbb{R}^n$ is an operator of the form

$$Pf(x) = \int_{\mathbb{R}^n} a(x,\xi)\hat{f}(\xi)e^{i\langle \xi, x\rangle}\, d\xi,$$

where $a(x,\xi)$ is a **polyhomog. symbol** (generalizing a polynomial in ξ).

Theorem (Couchet-Y., following Van Erp-Y.)

There is a 1–1 correspondence

$$\left\{ \begin{matrix} \textit{polyhomog. symbols on} \\ \mathbb{R}^n \times \mathbb{R}^n \textit{ of order} \leqslant m \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \textit{smooth functions on} \\ \mathbb{R}^n \times \mathbb{R}^{n+1} \textit{ homog. of} \\ \textit{degree} = m \textit{ mod Schwartz} \end{matrix} \right\}$$

$$a(x,\xi) = \underline{a}(x,\xi,1) \longleftarrow \underline{a}(x,\xi,t)$$

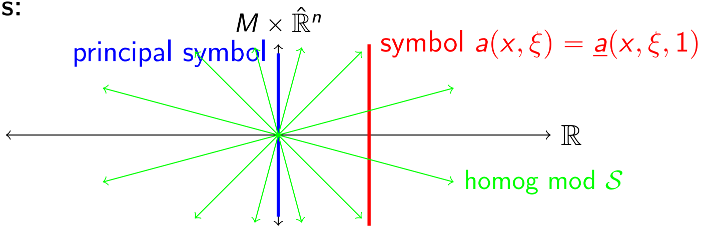
Homog. mod Schwartz means

$$\underline{a}(x,s\xi,st) - s^m \underline{a}(x,\xi,t) \in C^\infty(\mathbb{R}^n, \mathcal{S}(\mathbb{R}^{n+1})) \qquad \forall s > 0.$$

Symbols & kernels as slices at $t = 1$

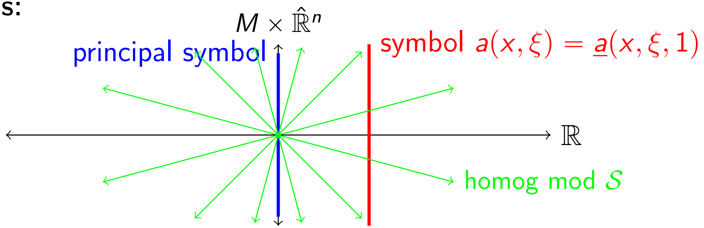
Symbols & kernels as slices at $t = 1$

Symbols:



Symbols & kernels as slices at $t = 1$

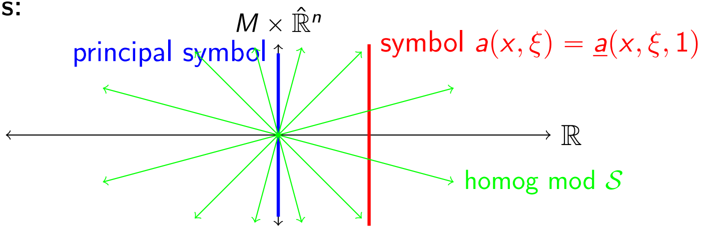
Symbols:



↓ Fourier transform in ξ

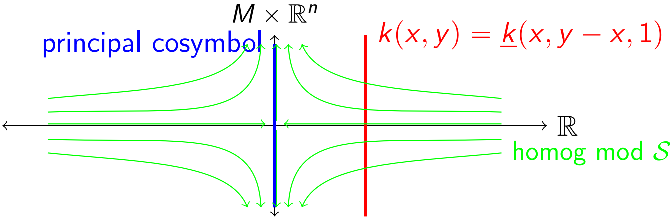
Symbols & kernels as slices at $t = 1$

Symbols:



↓ **Fourier transform in** ξ

Kernels:

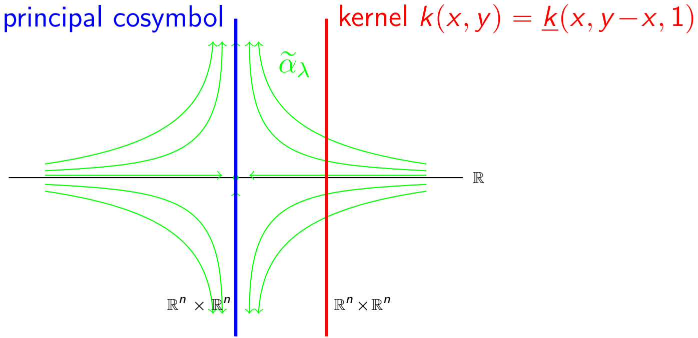


The groupoid approach to classical ΨDOs

This definition is **coordinate independent**. . .

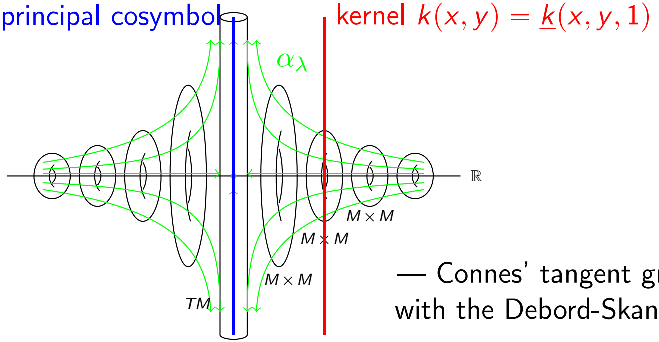
The groupoid approach to classical ΨDOs

This definition is **coordinate independent**. . .



The groupoid approach to classical ΨDOs

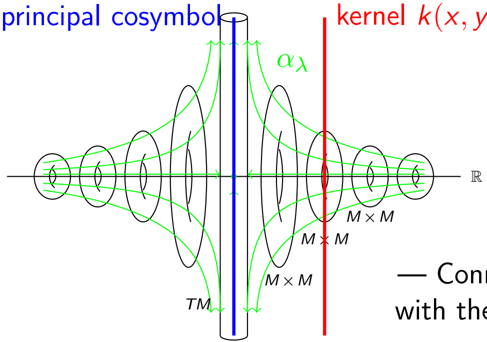
This definition is **coordinate independent**. . .



— Connes' tangent groupoid $\mathbb{T}M$ with the Debord-Skandalis action.

The groupoid approach to classical ΨDOs

This definition is **coordinate independent**. . .

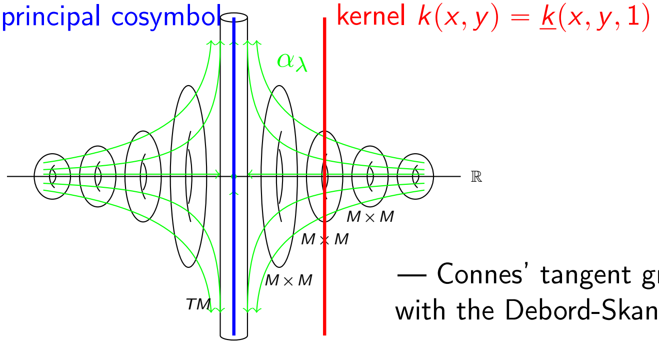


— Connes' tangent groupoid $\mathbb{T}M$ with the Debord-Skandalis action.

$$\mathbb{T}M = (M \times M \times \mathbb{R}^\times) \sqcup (TM \times \{0\})$$

The groupoid approach to classical ΨDOs

This definition is **coordinate independent**. . .



— Connes' tangent groupoid $\mathbb{T}M$ with the Debord-Skandalis action.

$$\mathbb{T}M = (M \times M \times \mathbb{R}^\times) \sqcup (TM \times \{0\})$$

Topology: $(x_i, y_i, t_i) \to (V_x, 0)$ iff $t_i \to 0$ and $\frac{x_i - y_i}{t_i} \to V_x$.

The groupoid approach to classical ΨDOs

Definition (Van Erp-Y., inspired by Debord-Skandalis)

A (properly supported, classical polyhomogeneous) ΨDO of order $\leqslant m$ on M is an operator with Schwartz kernel $k(x,y) = \underline{k}(x,y,1)$, where

$$\underline{k}(x,y,t) \in \mathcal{E}'_r(\mathbb{T}M) \qquad \text{[Lescure-Manchon-Vassout]}$$

is homogeneous of degree m modulo C^∞_{p} for the Debord-Skandalis action:

$$\alpha_s : \begin{cases} (x,y,t) = (x,y,s^{-1}t), & t \neq 0, \\ (x,v,0) = (x,sv,0), & \text{t=0.} \end{cases}$$

The groupoid approach to classical ΨDOs

Definition (Van Erp-Y., inspired by Debord-Skandalis)

A (properly supported, classical polyhomogeneous) ΨDO of order $\leqslant m$ on M is an operator with Schwartz kernel $k(x,y) = \underline{k}(x,y,1)$, where

$$\underline{k}(x,y,t) \in \mathcal{E}'_r(\mathbb{T}M)$$ [Lescure-Manchon-Vassout]

is homogeneous of degree m modulo C^∞_{p} for the Debord-Skandalis action:

$$\alpha_s : \begin{cases} (x,y,t) = (x,y,s^{-1}t), & t \neq 0, \\ (x,v,0) = (x,sv,0), & \text{t=0.} \end{cases}$$

Theorem (Van Erp-Y.)

1. *This is equivalent to the usual definition of classical ΨDOs [Kohn-Nirenberg].*
2. *The restriction $\underline{k}|_{t=0}$ is the principal cosymbol.*

Groupoid convolution

Theorem (Lescure-Manchon-Vassout)

The product of the Lie groupoid $\mathbb{T}M$ integrates to a ***convolution product*** *of r-fibred distributions on $\mathbb{T}M$,*

$$u * v(\gamma) = \int_{\beta \in G^{r(\gamma)}} u(\beta) v(\beta^{-1}\gamma) d\beta.$$

Groupoid convolution

Theorem (Lescure-Manchon-Vassout)

The product of the Lie groupoid $\mathbb{T}M$ integrates to a ***convolution product*** *of r-fibred distributions on $\mathbb{T}M$,*

$$u * v(\gamma) = \int_{\beta \in G^{r(\gamma)}} u(\beta) v(\beta^{-1}\gamma) d\beta.$$

Examples

1. **Pair groupoid** $M \times M$: $(x,z) = \quad (x,y) \ (y,z)$

Groupoid convolution

Theorem (Lescure-Manchon-Vassout)

The product of the Lie groupoid $\mathbb{T}M$ integrates to a ***convolution product*** *of r-fibred distributions on $\mathbb{T}M$,*

$$u * v(\gamma) = \int_{\beta \in G^{r(\gamma)}} u(\beta) v(\beta^{-1}\gamma) d\beta.$$

Examples

1. **Pair groupoid $M \times M$:** $a * b(x,z) = \int_{y \in M} a(x,y) b(y,z)\, dy$

   — composition of Schwartz kernels.

Groupoid convolution

Theorem (Lescure-Manchon-Vassout)

The product of the Lie groupoid $\mathbb{T}M$ integrates to a ***convolution product*** *of r-fibred distributions on $\mathbb{T}M$,*

$$u * v(\gamma) = \int_{\beta \in G^{r(\gamma)}} u(\beta) v(\beta^{-1}\gamma) d\beta.$$

Examples

1. **Pair groupoid $M \times M$:** $a * b(x, z) = \int_{y \in M} a(x, y) b(y, z)\, dy$

   — composition of Schwartz kernels.

2. **Tangent bundle TM:** $v_x = (v_p - w_p) + (w_p)$

Groupoid convolution

Theorem (Lescure-Manchon-Vassout)

The product of the Lie groupoid $\mathbb{T}M$ integrates to a ***convolution product*** *of r-fibred distributions on $\mathbb{T}M$,*

$$u * v(\gamma) = \int_{\beta \in G^{r(\gamma)}} u(\beta) v(\beta^{-1}\gamma) d\beta.$$

Examples

1. **Pair groupoid $M \times M$:** $a * b(x,z) = \int_{y \in M} a(x,y) b(y,z)\, dy$

   — composition of Schwartz kernels.

2. **Tangent bundle TM:** $a * b(v_x) = \int_{w \in TM_x} a(v_p - w_p)\, b(w_p)\, dw$

   — (Fourier transform of) product of principal symbols.

Groupoid convolution

Theorem (Lescure-Manchon-Vassout)

The product of the Lie groupoid $\mathbb{T}M$ integrates to a ***convolution product*** *of r-fibred distributions on $\mathbb{T}M$,*

$$u * v(\gamma) = \int_{\beta \in G^{r(\gamma)}} u(\beta) v(\beta^{-1}\gamma) d\beta.$$

Examples

1. **Pair groupoid** $M \times M$: $\quad a * b(x, z) = \int_{y \in M} a(x, y) b(y, z)\, dy$

   — composition of Schwartz kernels.
2. **Tangent bundle** TM: $\quad a * b(v_x) = \int_{w \in TM_x} a(v_p - w_p)\, b(w_p)\, dw$

   — (Fourier transform of) product of principal symbols.

$\Rightarrow$ Groupoid convolution in $\mathcal{E}'_r(\mathbb{T}M)$ describes both composition of ΨDOs (at $t = 1$) and product of symbols (at $t = 0$). [Connes]

Elliptic $\Rightarrow$ Hypoelliptic

Definition

A differential operator $P \in \mathrm{DO}^m(M)$ is **elliptic** if its principal symbol is invertible outside 0 for pointwise product.

Elliptic ⇒ Hypoelliptic

Definition

A differential operator $P \in \mathrm{DO}^m(M)$ is **elliptic** if its principal cosymbol is invertible modulo Schwartz for fibrewise convolution product.

Elliptic $\Rightarrow$ Hypoelliptic

Definition

A differential operator $P \in \mathrm{DO}^m(M)$ is **elliptic** if its principal cosymbol is invertible modulo Schwartz for fibrewise convolution product.

Easy to check that $\Psi^m(M)$ has the three keys properties of a ΨD calculus:

1. Regularity
2. Pseudolocality
3. Parametrices of elliptic elements.

Elliptic $\Rightarrow$ Hypoelliptic

Definition

A differential operator $P \in \mathrm{DO}^m(M)$ is **elliptic** if its principal cosymbol is invertible modulo Schwartz for fibrewise convolution product.

Easy to check that $\Psi^m(M)$ has the three keys properties of a ΨD calculus:

1. Regularity
2. Pseudolocality
3. Parametrices of elliptic elements.

Corollary

P elliptic $\Rightarrow$ P hypoelliptic.

A calculus for Helffer-Nourrigat operators

Plan

Plan

Goal

Generalize the groupoid calculus to Helffer-Nourrigat operators.

Plan

Goal

Generalize the groupoid calculus to Helffer-Nourrigat operators.

1. Construct an appropriate tangent groupoid $\mathbb{T}\mathcal{F}$.
   (using Androulidakis-Skandalis' singular holonomy groupoid).

Plan

Goal

Generalize the groupoid calculus to Helffer-Nourrigat operators.

1. Construct an appropriate tangent groupoid $\mathbb{T}\mathcal{F}$.
   (using Androulidakis-Skandalis' singular holonomy groupoid).
2. Define $\Psi^m(M)$ as above: Restriction at $t = 1$ of homogeneous distributions on $\mathbb{T}\mathcal{F}$ modulo C^∞.
   $\Rightarrow$ pseudolocality, regularity.

Plan

Goal

Generalize the groupoid calculus to Helffer-Nourrigat operators.

1. Construct an appropriate tangent groupoid $\mathbb{T}\mathcal{F}$.
   (using Androulidakis-Skandalis' singular holonomy groupoid).
2. Define $\Psi^m(M)$ as above: Restriction at $t = 1$ of homogeneous distributions on $\mathbb{T}\mathcal{F}$ modulo C^∞.
   $\Rightarrow$ pseudolocality, regularity.
3. Make a precise notion of "ellipticity": Helffer-Nourrigat condition.
   Mohsen's blow-up groupoid $\Rightarrow$ parametrices.

Plan

Goal

Generalize the groupoid calculus to Helffer-Nourrigat operators.

1. Construct an appropriate tangent groupoid $\mathbb{T}\mathcal{F}$.
   (using Androulidakis-Skandalis' singular holonomy groupoid).
2. Define $\Psi^m(M)$ as above: Restriction at $t = 1$ of homogeneous distributions on $\mathbb{T}\mathcal{F}$ modulo C^∞.
   $\Rightarrow$ pseudolocality, regularity.
3. Make a precise notion of "ellipticity": Helffer-Nourrigat condition.
   Mohsen's blow-up groupoid $\Rightarrow$ parametrices.

Corollary (The Helffer-Nourrigat Conjecture)

Helffer-Nourrigat operators are (maximally) hypoelliptic.

The singular tangent groupoid

Families of vector fields

M — manifold

$V_1, \ldots, V_n$ — vector fields on M.

Families of vector fields

M — manifold

$V_1, \ldots, V_n$ — vector fields on M.

$\mathcal{F}^k := C^\infty(M)$-module gen. by iterated Lie brackets of V_i's of length $\leqslant i$.

Families of vector fields

M — manifold

$V_1, \ldots, V_n$ — vector fields on M.

$\mathcal{F}^k := C^\infty(M)$-module gen. by iterated Lie brackets of V_i's of length $\leqslant i$.

By construction, $[\mathcal{F}^j, \mathcal{F}^k] \subseteq \mathcal{F}^{j+k}$.

Families of vector fields

M — manifold

$V_1, \ldots, V_n$ — vector fields on M.

$\mathcal{F}^k := C^\infty(M)$-module gen. by iterated Lie brackets of V_i's of length $\leqslant i$.

By construction, $[\mathcal{F}^j, \mathcal{F}^k] \subseteq \mathcal{F}^{j+k}$.
$\Rightarrow \mathcal{F} = \bigcup_{k \leqslant 1} \mathcal{F}^k$ is a filtered Lie algebra of vector fields.

Families of vector fields

M — manifold

$V_1, \ldots, V_n$ — vector fields on M.

$\mathcal{F}^k := C^\infty(M)$-module gen. by iterated Lie brackets of V_i's of length $\leqslant i$.

By construction, $[\mathcal{F}^j, \mathcal{F}^k] \subseteq \mathcal{F}^{j+k}$.
$\Rightarrow \mathcal{F} = \bigcup_{k \leqslant 1} \mathcal{F}^k$ is a filtered Lie algebra of vector fields.

Definition

- If $\mathcal{F}^N = \mathcal{F}^{N+1} = \cdots$ for some N, we say $\mathcal{F}$ is a **filtered singular foliation** of depth N.
- In particular, if $\mathcal{F}^N = \Gamma^\infty(TM)$, we say the family $(V_1, \ldots, V_d)$ satisfies **Hörmander's bracket-generating condition**.

Families of vector fields

M — manifold

$V_1, \dots, V_n$ — vector fields on M.

$\mathcal{F}^k := C^\infty(M)$-module gen. by iterated Lie brackets of V_i's of length $\leqslant i$.

By construction, $[\mathcal{F}^j, \mathcal{F}^k] \subseteq \mathcal{F}^{j+k}$.
$\Rightarrow \mathcal{F} = \bigcup_{k \leqslant 1} \mathcal{F}^k$ is a filtered Lie algebra of vector fields.

Definition

- If $\mathcal{F}^N = \mathcal{F}^{N+1} = \cdots$ for some N, we say $\mathcal{F}$ is a **filtered singular foliation** of depth N.
- In particular, if $\mathcal{F}^N = \Gamma^\infty(TM)$, we say the family $(V_1, \dots, V_d)$ satisfies **Hörmander's bracket-generating condition**.

NB. More general filtered singular foliations are possible, eg, by ascribing an *order* to each V_i.

The osculating groups

Not'n. $I_p := \{f \in C^\infty(M) \mid f(p) = 0\}$ — vanishing ideal.

$\mathcal{F}^k_p := \mathcal{F}^k / I_p \mathcal{F}^k$ — fibre of $\mathcal{F}^k$ at p.

The osculating groups

Not'n. $I_p := \{f \in C^\infty(M) \mid f(p) = 0\}$ — vanishing ideal.

$\mathcal{F}^k_p := \mathcal{F}^k / I_p \mathcal{F}^k$ — fibre of $\mathcal{F}^k$ at p.

Definition (Osculating groups)

The **osculating Lie algebra** of $\mathcal{F}$ at p is the "associated graded of $\mathcal{F}_p$",

$$\mathfrak{gr}\mathcal{F}_p := \bigoplus_k \mathfrak{gr}_k \mathcal{F}_p,$$

where $\mathfrak{gr}_k \mathcal{F}_p := \mathcal{F}^k_p / \mathcal{F}^{k-1}_p = \mathcal{F}^k / (\mathcal{F}^{k-1} + I_p \mathcal{F}^k)$.

The osculating groups

Not'n. $I_p := \{f \in C^\infty(M) \mid f(p) = 0\}$ — vanishing ideal.

$\mathcal{F}^k_p := \mathcal{F}^k / I_p \mathcal{F}^k$ — fibre of $\mathcal{F}^k$ at p.

Definition (Osculating groups)

The **osculating Lie algebra** of $\mathcal{F}$ at p is the "associated graded of $\mathcal{F}_p$",

$$\mathfrak{gr}\mathcal{F}_p := \bigoplus_k \mathfrak{gr}_k \mathcal{F}_p,$$

where $\mathfrak{gr}_k \mathcal{F}_p := \mathcal{F}^k_p / \mathcal{F}^{k-1}_p = \mathcal{F}^k / (\mathcal{F}^{k-1} + I_p \mathcal{F}^k)$.

- [!] In this generality, the "inclusion" $\mathcal{F}^{k-1}_p \to \mathcal{F}^k_p$ may not be injective.
  $\Rightarrow$ may have dimension jumps: $\dim \mathfrak{gr}\mathcal{F}_p > \dim \mathcal{F}_p$.

The osculating groups

Not'n. $I_p := \{f \in C^\infty(M) \mid f(p) = 0\}$ — vanishing ideal.

$\mathcal{F}^k_p := \mathcal{F}^k / I_p \mathcal{F}^k$ — fibre of $\mathcal{F}^k$ at p.

Definition (Osculating groups)

The **osculating Lie algebra** of $\mathcal{F}$ at p is the "associated graded of $\mathcal{F}_p$",

$$\mathfrak{gr}\mathcal{F}_p := \bigoplus_k \mathfrak{gr}_k \mathcal{F}_p,$$

where $\mathfrak{gr}_k \mathcal{F}_p := \mathcal{F}^k_p / \mathcal{F}^{k-1}_p = \mathcal{F}^k / (\mathcal{F}^{k-1} + I_p \mathcal{F}^k)$.

- [!] In this generality, the "inclusion" $\mathcal{F}^{k-1}_p \to \mathcal{F}^k_p$ may not be injective.
  $\Rightarrow$ may have dimension jumps: $\dim \mathfrak{gr}\mathcal{F}_p > \dim \mathcal{F}_p$.
- Filtered Lie alg. structure on $\mathcal{F}$ $\rightsquigarrow$ Graded Lie alg. structure on $\mathfrak{gr}\mathcal{F}_p$.

The osculating groups

Not'n. $I_p := \{f \in C^\infty(M) \mid f(p) = 0\}$ — vanishing ideal.

$\mathcal{F}_p^k := \mathcal{F}^k / I_p \mathcal{F}^k$ — fibre of $\mathcal{F}^k$ at p.

Definition (Osculating groups)

The **osculating Lie algebra** of $\mathcal{F}$ at p is the "associated graded of $\mathcal{F}_p$",

$$\mathfrak{gr}\mathcal{F}_p := \bigoplus_k \mathfrak{gr}_k \mathcal{F}_p,$$

where $\mathfrak{gr}_k \mathcal{F}_p := \mathcal{F}_p^k / \mathcal{F}_p^{k-1} = \mathcal{F}^k / (\mathcal{F}^{k-1} + I_p \mathcal{F}^k)$.

The **osculating Lie group** is $\mathrm{Gr}\mathcal{F}_p = \mathfrak{gr}\mathcal{F}_p$ with BCH product.

- [!] In this generality, the "inclusion" $\mathcal{F}_p^{k-1} \to \mathcal{F}_p^k$ may not be injective.
  $\Rightarrow$ may have dimension jumps: $\dim \mathfrak{gr}\mathcal{F}_p > \dim \mathcal{F}_p$.
- Filtered Lie alg. structure on $\mathcal{F}$ $\rightsquigarrow$ Graded Lie alg. structure on $\mathfrak{gr}\mathcal{F}_p$.

Example: A hypoelliptic Laplacian

$P = \partial_x^2 + x\partial_y$ on $M = \mathbb{R}^2$.

Example: A hypoelliptic Laplacian

$P = \partial_x^2 + x\partial_y$ on $M = \mathbb{R}^2$.

- $X = \partial_x$, $Y = x\partial_y$, $Z = [X, Y] = \partial_y$.

Example: A hypoelliptic Laplacian

$P = \partial_x^2 + x\partial_y$ on $M = \mathbb{R}^2$.

- $X = \partial_x,\ Y = x\partial_y,\ Z = [X, Y] = \partial_y$.
- $\mathcal{F}^1 = \langle X \rangle \quad \subseteq \quad \mathcal{F}^2 = \langle X, Y \rangle \quad \subseteq \quad \mathcal{F}^3 = \langle X, Y, Z \rangle = \Gamma^\infty(M)$
  (depth 3)

Example: A hypoelliptic Laplacian

$P = \partial_x^2 + x\partial_y$ on $M = \mathbb{R}^2$.

- $X = \partial_x$, $Y = x\partial_y$, $Z = [X, Y] = \partial_y$.
- $\mathcal{F}^1 = \langle X \rangle \subseteq \mathcal{F}^2 = \langle X, Y \rangle \subseteq \mathcal{F}^3 = \langle X, Y, Z \rangle = \Gamma^\infty(M)$ (depth 3)
- In a neighbourhood of $p = (x, y)$:
  If $x \neq 0$ then $Z \in \mathcal{F}^2_p \Rightarrow [Z]_p = 0 \in \mathfrak{gr}_3\mathcal{F}_p$,
  If $x = 0$ then $Z \notin \mathcal{F}^2_p \Rightarrow [Z]_p \neq 0 \in \mathfrak{gr}_3\mathcal{F}_p$.

Example: A hypoelliptic Laplacian

$P = \partial_x^2 + x\partial_y$ on $M = \mathbb{R}^2$.

- $X = \partial_x$, $Y = x\partial_y$, $Z = [X, Y] = \partial_y$.
- $\mathcal{F}^1 = \langle X \rangle \subseteq \mathcal{F}^2 = \langle X, Y \rangle \subseteq \mathcal{F}^3 = \langle X, Y, Z \rangle = \Gamma^\infty(M)$ (depth 3)
- In a neighbourhood of $p = (x, y)$:
  If $x \neq 0$ then $Z \in \mathcal{F}_p^2 \Rightarrow [Z]_p = 0 \in \mathfrak{gr}_3\mathcal{F}_p$,
  If $x = 0$ then $Z \notin \mathcal{F}_p^2 \Rightarrow [Z]_p \neq 0 \in \mathfrak{gr}_3\mathcal{F}_p$.

Osculating Lie algebras

$$\mathfrak{gr}\mathcal{F}_p = \begin{cases} \langle [X]_p \rangle \oplus \langle [Y]_p \rangle \cong \mathbb{R}^2, & \text{if } x \neq 0, \\ \langle [X]_p \rangle \oplus \langle [Y]_p \rangle \oplus \langle [Z]_p \rangle \cong \mathfrak{h}^3, & \text{if } x = 0, \end{cases}$$

Singular tangent groupoid (low-brow definition)

Singular tangent groupoid (low-brow definition)

Definition

- The **pair groupoid** is $M \times M$
  with partially defined product $(p,q)(q,r) = (p,r)$

Singular tangent groupoid (low-brow definition)

Definition

- The **pair groupoid** is $M \times M$
  with partially defined product $(p, q)(q, r) = (p, r)$
- The **osculating groupoid** of $\mathcal{F}$ is $\mathrm{Gr}\mathcal{F} := \bigsqcup_p \mathrm{Gr}\mathcal{F}_p$
  with product defined only in each fibre

Singular tangent groupoid (low-brow definition)

Definition

- The **pair groupoid** is $M \times M$ with partially defined product $(p, q)(q, r) = (p, r)$
- The **osculating groupoid** of $\mathcal{F}$ is $\mathrm{Gr}\mathcal{F} := \bigsqcup_p \mathrm{Gr}\mathcal{F}_p$ with product defined only in each fibre
- The **tangent groupoid** of $\mathcal{F}$ is the family over $\mathbb{R}$:
$$\mathbb{T}\mathcal{F} := (M \times M \times \mathbb{R}^\times) \sqcup (\mathrm{Gr}\mathcal{F} \times \{0\}).$$

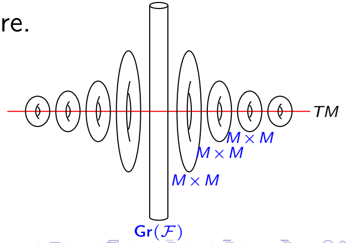
Singular tangent groupoid (low-brow definition)

Definition

- The **pair groupoid** is $M \times M$
  with partially defined product $(p,q)(q,r) = (p,r)$
- The **osculating groupoid** of $\mathcal{F}$ is $\mathrm{Gr}\mathcal{F} := \bigsqcup_p \mathrm{Gr}\mathcal{F}_p$
  with product defined only in each fibre
- The **tangent groupoid** of $\mathcal{F}$ is the family over $\mathbb{R}$:

$$\mathbb{T}\mathcal{F} := (M \times M \times \mathbb{R}^\times) \sqcup (\mathrm{Gr}\mathcal{F} \times \{0\}).$$

...but this doesn't explain the smooth structure.



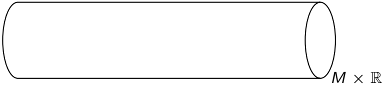
Singular tangent groupoid (high-brow definition)

[Androulidakis-Skandalis]

Singular tangent groupoid (high-brow definition)

[Androulidakis-Skandalis]

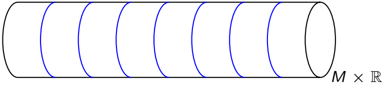
Given $\mathcal{F}$ a filtered sing. foliation on M, extend it to $M \times \mathbb{R}$ with speed change. . .



Singular tangent groupoid (high-brow definition)

[Androulidakis-Skandalis]

Given $\mathcal{F}$ a filtered sing. foliation on M, extend it to $M \times \mathbb{R}$ with speed change...

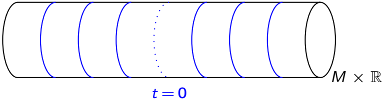


- $\widetilde{\mathcal{F}} := \{X \in \mathfrak{X}(M \times \mathbb{R}) \mid X|_{M \times \{t\}} \in \mathcal{F} \quad \forall t\}$

Singular tangent groupoid (high-brow definition)

[Androulidakis-Skandalis]

Given $\mathcal{F}$ a filtered sing. foliation on M, extend it to $M \times \mathbb{R}$ with speed change...

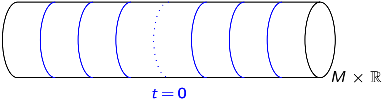


- $\widetilde{\mathcal{F}} := \{X \in \mathfrak{X}(M \times \mathbb{R}) \mid X|_{M \times \{t\}} \in \mathcal{F} \quad \forall t\}$
- The adiabatic foliation is $\mathfrak{a}\mathcal{F} = t\widetilde{\mathcal{F}}^1 + \cdots + t^N \widetilde{\mathcal{F}}^N$.

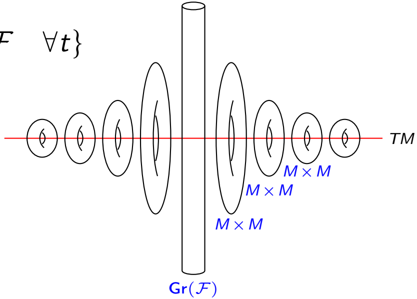
Singular tangent groupoid (high-brow definition)

[Androulidakis-Skandalis]

Given $\mathcal{F}$ a filtered sing. foliation on M, extend it to $M \times \mathbb{R}$ with speed change...



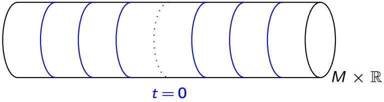
- $\widetilde{\mathcal{F}} := \{X \in \mathfrak{X}(M \times \mathbb{R}) \mid X|_{M\times\{t\}} \in \mathcal{F} \quad \forall t\}$
- The adiabatic foliation is $\mathfrak{a}\mathcal{F} = t\widetilde{\mathcal{F}}^1 + \cdots + t^N\widetilde{\mathcal{F}}^N$.
- The tangent groupoid is the singular holonomy groupoid $$\mathbb{T}\mathcal{F} := \mathcal{H}(\mathfrak{a}\mathcal{F})$$ of Androulidakis-Skandalis.



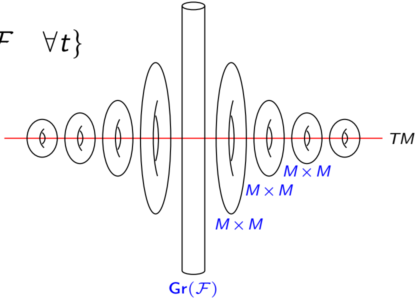
Singular tangent groupoid (high-brow definition)

[Androulidakis-Skandalis]

Given $\mathcal{F}$ a filtered sing. foliation on M, extend it to $M \times \mathbb{R}$ with speed change...



- $\widetilde{\mathcal{F}} := \{X \in \mathfrak{X}(M \times \mathbb{R}) \mid X|_{M \times \{t\}} \in \mathcal{F} \quad \forall t\}$
- The adiabatic foliation is $\mathfrak{a}\mathcal{F} = t\widetilde{\mathcal{F}}^1 + \cdots + t^N \widetilde{\mathcal{F}}^N$.
- The tangent groupoid is the singular holonomy groupoid
$$\mathbb{T}\mathcal{F} := \mathcal{H}(\mathfrak{a}\mathcal{F})$$
of Androulidakis-Skandalis.



It is a non-Hausdorff groupoid, but its r- & s-fibres are manifolds [Debord].

Smooth structure (low-brow version)

Smooth structure (low-brow version)

- Tangent algebroid $\mathfrak{t}\mathcal{F} := \quad (TM \times \mathbb{R}^\times) \quad \sqcup (\mathfrak{gr}\mathcal{F} \times \{0\})$

  — tangent bundle to r-fibres along base.

Smooth structure (low-brow version)

- Tangent algebroid $\mathfrak{t}\mathcal{F} := \quad (TM \times \mathbb{R}^\times) \quad \sqcup (\mathfrak{gr}\mathcal{F} \times \{0\})$

  — tangent bundle to r-fibres along base.

- Define a (not necessarily Hausdorff) vector bundle topology on $\mathfrak{t}\mathcal{F}$ such that the following sections are continuous: $\forall X \in \mathcal{F}^k$

$$(p,t) \mapsto \begin{cases} t^k X(p) & \in TM_p, \quad t \neq 0, \\ [X]_p & \in \mathfrak{gr}_k \mathcal{F}_p, \quad t = 0. \end{cases}$$

Smooth structure (low-brow version)

- Tangent algebroid $\mathfrak{t}\mathcal{F} := \quad (TM \times \mathbb{R}^\times) \quad \sqcup (\mathfrak{gr}\mathcal{F} \times \{0\})$

  — tangent bundle to r-fibres along base.

- Define a (not necessarily Hausdorff) vector bundle diffeology on $\mathfrak{t}\mathcal{F}$ such that the following sections are smooth: $\forall X \in \mathcal{F}^k$

$$(p,t) \mapsto \begin{cases} t^k X(p) & \in TM_p, \quad t \neq 0, \\ [X]_p & \in \mathfrak{gr}_k \mathcal{F}_p, \quad t = 0. \end{cases}$$

Smooth structure (low-brow version)

- Tangent algebroid $\mathfrak{t}\mathcal{F} := \quad (TM \times \mathbb{R}^\times) \quad \sqcup (\mathfrak{gr}\mathcal{F} \times \{0\})$

  — tangent bundle to r-fibres along base.

- Define a (not necessarily Hausdorff) vector bundle diffeology on $\mathfrak{t}\mathcal{F}$ such that the following sections are smooth: $\forall X \in \mathcal{F}^k$

$$(p,t) \mapsto \begin{cases} t^k X(p) & \in TM_p, & t \neq 0, \\ [X]_p & \in \mathfrak{gr}_k\mathcal{F}_p, & t = 0. \end{cases}$$

- This induces a smooth structure on the tangent groupoid $\mathbb{T}\mathcal{F}$ by a tubular neighbourhood construction (exponential charts).

ΨDOs & the Helffer-Nourrigat Conjecture

The $\mathbb{R}_+^\times$-action

- The osculating groups $\mathfrak{gr}\mathcal{F}_p$ admit **dilations** δ_s, given by multiplication by s^k on $\mathfrak{gr}_k\mathcal{F}_p$.

The $\mathbb{R}_+^\times$-action

- The osculating groups $\mathfrak{gr}\mathcal{F}_p$ admit **dilations** δ_s, given by multiplication by s^k on $\mathfrak{gr}_k\mathcal{F}_p$.
- The Debord-Skandalis action $\alpha_s : \mathbb{T}\mathcal{F} \to \mathbb{T}\mathcal{F}$,

$$\alpha_s : \begin{cases} (p,q,t) \to (p,q,s^{-1}t), & t \neq 0 \\ (p,v,t) \to (p,\delta_s(v),t), & t = 0 \end{cases}$$

is an action by smooth groupoid automorphisms.

The $\mathbb{R}_+^\times$-action

- The osculating groups $\mathfrak{gr}\mathcal{F}_p$ admit **dilations** δ_s, given by multiplication by s^k on $\mathfrak{gr}_k\mathcal{F}_p$.
- The Debord-Skandalis action $\alpha_s : \mathbb{T}\mathcal{F} \to \mathbb{T}\mathcal{F}$,

$$\alpha_s : \begin{cases} (p,q,t) \to (p,q,s^{-1}t), & t \neq 0 \\ (p,v,t) \to (p,\delta_s(v),t), & t = 0 \end{cases}$$

is an action by smooth groupoid automorphisms.

Definition

A distribution $\underline{k} \in \mathcal{E}'_r(\mathbb{T}\mathcal{F})$ is **essentially homogeneous of degree** m if

$$\alpha_{s*}\underline{k} - s^m \underline{k} \in C^\infty_{\mathrm{p}}(\mathbb{T}\mathcal{F}) \qquad \forall s > 0.$$

Pseudodifferential operators

Henceforth, suppose the bracket-generating condition: $\mathcal{F}^N = \Gamma^\infty(M)$.

Pseudodifferential operators

Henceforth, suppose the bracket-generating condition: $\mathcal{F}^N = \Gamma^\infty(M)$.

Definition

- A **pseudodifferential operator adapted to** $\mathcal{F}$ is an operator with Schwartz kernel $k = \underline{k}|_{t=1}$, where $\underline{k} \in \mathcal{E}'_r(\mathbb{T}\mathcal{F})$ is ess. homogeneous of degree m.
- The set of such ΨDOs is denored $\Psi^m(\mathcal{F})$.

Pseudodifferential operators

Henceforth, suppose the bracket-generating condition: $\mathcal{F}^N = \Gamma^\infty(M)$.

Definition

- A **pseudodifferential operator adapted to** $\mathcal{F}$ is an operator with Schwartz kernel $k = \underline{k}|_{t=1}$, where $\underline{k} \in \mathcal{E}'_r(\mathbb{T}\mathcal{F})$ is ess. homogeneous of degree m.
- The set of such ΨDOs is denored $\Psi^m(\mathcal{F})$.

It is easy to show that $\Psi^m(\mathcal{F})$ is a $\mathbb{Z}$-filtered algebra satisfying

1. Regularity
2. Pseudolocality

The question of parametrices of "elliptic" elements is far more subtle.

[Helffer-Nourrigat]

Topology of the dual bundle $\mathfrak{t}^*\mathcal{F}$

Definition (Local generating family of vector fields)

A family of vector fields $(X_1, \ldots, X_d)$ is a **local generating family** for $\mathcal{F}$ on $U \subseteq M$ if there are $d_1, \ldots, d_N$ s.t. $(X_1, \ldots, X_{d_k})$ spans $\mathcal{F}^k$ on U.

Topology of the dual bundle $\mathfrak{t}^*\mathcal{F}$

Definition (Local generating family of vector fields)

A family of vector fields $(X_1, \ldots, X_d)$ is a **local generating family** for $\mathcal{F}$ on $U \subseteq M$ if there are $d_1, \ldots, d_N$ s.t. $(X_1, \ldots, X_{d_k})$ spans $\mathcal{F}^k$ on U.

- The smooth "charts" of $\mathfrak{t}\mathcal{F}$,

$$\begin{array}{rcl} U \times \mathbb{R}^d \times \mathbb{R} & \to & \mathfrak{t}\mathcal{F} \\ (p, \xi, t) & \mapsto & \begin{cases} (p, \sum_i \xi_i t^{a_i} X_i(p), t), & t \neq 0 \quad (a_i = \mathrm{ord}(X_i)) \\ (p, \ \sum_i \xi_i [X_i]_p, \ 0), & t = 0, \end{cases} \end{array}$$

are locally surjective bundle maps.

Topology of the dual bundle $\mathfrak{t}^*\mathcal{F}$

Definition (Local generating family of vector fields)

A family of vector fields $(X_1, \ldots, X_d)$ is a **local generating family** for $\mathcal{F}$ on $U \subseteq M$ if there are $d_1, \ldots, d_N$ s.t. $(X_1, \ldots, X_{d_k})$ spans $\mathcal{F}^k$ on U.

- The smooth "charts" of $\mathfrak{t}\mathcal{F}$,

$$
\begin{aligned}
U \times \mathbb{R}^d \times \mathbb{R} \quad &\rightarrow \quad \mathfrak{t}\mathcal{F} \\
(p, \xi, t) \quad &\mapsto \quad \begin{cases} (p, \sum_i \xi_i t^{a_i} X_i(p), t), & t \neq 0 \qquad (a_i = \operatorname{ord}(X_i)) \\ (p, \ \sum_i \xi_i [X_i]_p, \ 0), & t = 0, \end{cases}
\end{aligned}
$$

  are locally surjective bundle maps.

$\Rightarrow$ Dual maps $\mathfrak{t}^*\mathcal{F} \rightarrow U \times \mathbb{R}^{d*} \times \mathbb{R}$ are locally injective bundle maps.

Topology of the dual bundle $\mathfrak{t}^*\mathcal{F}$

Definition (Local generating family of vector fields)

A family of vector fields $(X_1, \dots, X_d)$ is a **local generating family** for $\mathcal{F}$ on $U \subseteq M$ if there are $d_1, \dots, d_N$ s.t. $(X_1, \dots, X_{d_k})$ spans $\mathcal{F}^k$ on U.

- The smooth "charts" of $\mathfrak{t}\mathcal{F}$,

$$\begin{array}{rcl} U \times \mathbb{R}^d \times \mathbb{R} & \to & \mathfrak{t}\mathcal{F} \\ (p, \xi, t) & \mapsto & \begin{cases} (p, \sum_i \xi_i t^{a_i} X_i(p), t), & t \neq 0 \quad (a_i = \mathrm{ord}(X_i)) \\ (p, \ \sum_i \xi_i [X_i]_p, \ 0), & t = 0, \end{cases} \end{array}$$

  are locally surjective bundle maps.

$\Rightarrow$ Dual maps $\mathfrak{t}^*\mathcal{F} \to U \times \mathbb{R}^{d*} \times \mathbb{R}$ are locally injective bundle maps.

$\Rightarrow$ $\mathfrak{t}^*\mathcal{F}$ is a locally compact Hausdorff bundle (non locally trivial), with $\mathfrak{t}^*\mathcal{F}|_{t>0} = T^*M$, but potential dimension jumps at $\mathfrak{t}^*\mathcal{F}|_{t=0}$.

Topology of the dual bundle $\mathfrak{t}^*\mathcal{F}$

Definition (Local generating family of vector fields)

A family of vector fields $(X_1, \ldots, X_d)$ is a **local generating family** for $\mathcal{F}$ on $U \subseteq M$ if there are $d_1, \ldots, d_N$ s.t. $(X_1, \ldots, X_{d_k})$ spans $\mathcal{F}^k$ on U.

- The smooth "charts" of $\mathfrak{t}\mathcal{F}$,

$$\begin{array}{rcl} U \times \mathbb{R}^d \times \mathbb{R} & \to & \mathfrak{t}\mathcal{F} \\ (p, \xi, t) & \mapsto & \begin{cases} (p, \sum_i \xi_i t^{a_i} X_i(p), t), & t \neq 0 \quad (a_i = \mathrm{ord}(X_i)) \\ (p, \ \sum_i \xi_i [X_i]_p, \ 0), & t = 0, \end{cases} \end{array}$$

  are locally surjective bundle maps.

$\Rightarrow$ Dual maps $\mathfrak{t}^*\mathcal{F} \to U \times \mathbb{R}^{d*} \times \mathbb{R}$ are locally injective bundle maps.

$\Rightarrow$ $\mathfrak{t}^*\mathcal{F}$ is a locally compact Hausdorff bundle (non locally trivial), with $\mathfrak{t}^*\mathcal{F}|_{t>0} = T^*M$, but potential dimension jumps at $\mathfrak{t}^*\mathcal{F}|_{t=0}$.

- [!] Not all of $\mathfrak{gr}\mathcal{F}^*_p = \mathfrak{t}^*\mathcal{F}|_{t=0}$ will be a limit of $\mathfrak{t}^*\mathcal{F}|_{t>0}$ as $t \to 0$.

Example: A hypoellipitic Laplacian

$P = \partial_x^2 + x\partial_y$ on $M = \mathbb{R}^2$.

Example: A hypoellipitic Laplacian

$P = \partial_x^2 + x\partial_y$ on $M = \mathbb{R}^2$.

- $X = \partial_x$, $Y = x\partial_y$, $Z = [X, Y] = \partial_y$.

Example: A hypoellipitic Laplacian

$P = \partial_x^2 + x\partial_y$ on $M = \mathbb{R}^2$.

- $X = \partial_x$, $Y = x\partial_y$, $Z = [X, Y] = \partial_y$.
- $\mathcal{F}^1 = \langle X \rangle \quad \subseteq \quad \mathcal{F}^2 = \langle X, Y \rangle \quad \subseteq \quad \mathcal{F}^3 = \langle X, Y, Z \rangle = \Gamma^\infty(M)$

Example: A hypoellipitic Laplacian

$P = \partial_x^2 + x\partial_y$ on $M = \mathbb{R}^2$.

- $X = \partial_x$, $Y = x\partial_y$, $Z = [X, Y] = \partial_y$.
- $\mathcal{F}^1 = \langle X \rangle \quad \subseteq \quad \mathcal{F}^2 = \langle X, Y \rangle \quad \subseteq \quad \mathcal{F}^3 = \langle X, Y, Z \rangle = \Gamma^\infty(M)$
- $\mathfrak{gr}\mathcal{F}_p = \begin{cases} \langle [X]_p \rangle \oplus \langle [Y]_p \rangle \cong \mathbb{R}^2, & \text{if } x \neq 0, \\ \langle [X]_p \rangle \oplus \langle [Y]_p \rangle \oplus \langle [Z]_p \rangle \cong \mathfrak{h}^3, & \text{if } x = 0, \end{cases}$

Example: A hypoellipitic Laplacian

$P = \partial_x^2 + x\partial_y$ on $M = \mathbb{R}^2$.

- $X = \partial_x$, $Y = x\partial_y$, $Z = [X, Y] = \partial_y$.
- $\mathcal{F}^1 = \langle X \rangle \quad \subseteq \quad \mathcal{F}^2 = \langle X, Y \rangle \quad \subseteq \quad \mathcal{F}^3 = \langle X, Y, Z \rangle = \Gamma^\infty(M)$
- $\mathfrak{gr}\mathcal{F}_p = \begin{cases} \langle [X]_p \rangle \oplus \langle [Y]_p \rangle & \cong \mathbb{R}^2, \quad \text{if } x \neq 0, \\ \langle [X]_p \rangle \oplus \langle [Y]_p \rangle \oplus \langle [Z]_p \rangle & \cong \mathfrak{h}^3, \quad \text{if } x = 0, \end{cases}$
- Chart $\mathbb{R}^3 \twoheadrightarrow \mathfrak{t}\mathcal{F}_{(x,y,t)}$ is

$$(a, b, c) \mapsto \begin{cases} ta\partial_x + (t^2xb + t^3c)\partial_y, & t \neq 0, \\ a[X]_p + b[Y]_p, & t = 0, \ x \neq 0, \\ a[X]_p + b[Y]_p + c[Z]_p, & t = 0, \ x = 0. \end{cases}$$

Example: A hypoellipitic Laplacian

$P = \partial_x^2 + x\partial_y$ on $M = \mathbb{R}^2$.

- $X = \partial_x$, $Y = x\partial_y$, $Z = [X, Y] = \partial_y$.
- $\mathcal{F}^1 = \langle X \rangle \quad \subseteq \quad \mathcal{F}^2 = \langle X, Y \rangle \quad \subseteq \quad \mathcal{F}^3 = \langle X, Y, Z \rangle = \Gamma^\infty(M)$
- $\mathfrak{gr}\mathcal{F}_p = \begin{cases} \langle [X]_p \rangle \oplus \langle [Y]_p \rangle \cong \mathbb{R}^2, & \text{if } x \neq 0, \\ \langle [X]_p \rangle \oplus \langle [Y]_p \rangle \oplus \langle [Z]_p \rangle \cong \mathfrak{h}^3, & \text{if } x = 0, \end{cases}$
- Chart $\mathbb{R}^3 \twoheadrightarrow \mathfrak{t}\mathcal{F}_{(x,y,t)}$ is

$$(a, b, c) \mapsto \begin{cases} ta\partial_x + (t^2xb + t^3c)\partial_y, & t \neq 0, \\ a[X]_p + b[Y]_p, & t = 0, \ x \neq 0, \\ a[X]_p + b[Y]_p + c[Z]_p, & t = 0, \ x = 0. \end{cases}$$

- Image of $\mathfrak{t}^*\mathcal{F}|_{(x,y,t)} \hookrightarrow \mathbb{R}^{3*}$ is:

$$\mathrm{span}\{(1,0,0), (0, t^2x, t^3)\} = (0, t, x)^\perp.$$

Example: Another hypoellipitic Laplacian

$P = \partial_x^2 + \frac{x^2}{2}\partial_y$ on $M = \mathbb{R}^2$.

- $X = \partial_x,\ Y = \frac{x^2}{2}\partial_y,\ Z = [X,Y] = x\partial_y,\ W = [X,Z] = \partial_y.$
- $\mathcal{F}^1 = \langle X \rangle \quad \subseteq \quad \mathcal{F}^2 = \langle X, Y \rangle \quad \subseteq \quad \mathcal{F}^3 = \langle X, Y, Z \rangle$
  $\subseteq \quad \mathcal{F}^4 = \langle X, Y, Z, W \rangle = \Gamma^\infty(M)$
- $\mathfrak{gr}\mathcal{F}_p = \begin{cases} \langle [X]_p \rangle \oplus \langle [Y]_p \rangle \cong \mathbb{R}^2, & \text{if } x \neq 0, \\ \langle [X]_p \rangle \oplus \langle [Y]_p \rangle \oplus \langle [Z]_p \rangle \oplus \langle [W]_p \rangle \cong \mathfrak{n}^4, & \text{if } x = 0, \end{cases}$
- Chart $\mathbb{R}^4 \twoheadrightarrow \mathfrak{t}\mathcal{F}_{(x,y,t)}$ is

$$(a,b,c,d) \mapsto \begin{cases} ta\partial_x + (\frac{1}{2}t^2x^2b + t^3xc + t^4d)\partial_y, & t \neq 0, \\ a[X]_p + b[Y]_p, & t = 0,\ x \neq 0, \\ a[X]_p + b[Y]_p + c[Z]_p + d[W]_p, & t = 0,\ x = 0. \end{cases}$$

- Image of dual $\mathfrak{t}^*\mathcal{F}|_{(x,y,t)}$ is:

$$\mathrm{span}\{(1,0,0), (0, \tfrac{1}{2}t^2x^2, t^3x, t^4)\}.$$

Principal cosymbols

Definition (Helffer-Nourrigat cone)

$$\mathrm{HN}(\mathcal{F}) := \overline{\mathfrak{t}^*\mathcal{F}|_{t>0}} \cap \mathfrak{t}^*\mathcal{F}|_{t=0}$$

Principal cosymbols

Definition (Helffer-Nourrigat cone)

$\mathrm{HN}(\mathcal{F}) := \overline{\mathfrak{t}^*\mathcal{F}|_{t>0}} \cap \mathfrak{t}^*\mathcal{F}|_{t=0}$

Examples

1. For $P = -\partial_x^2 + x\partial_y$, $\mathrm{HN}(\mathcal{F})_p = \mathfrak{t}^*\mathcal{F}_p$ for all $p \in M$.

Principal cosymbols

Definition (Helffer-Nourrigat cone)

$$\mathrm{HN}(\mathcal{F}) := \overline{\mathfrak{t}^*\mathcal{F}|_{t>0}} \cap \mathfrak{t}^*\mathcal{F}|_{t=0}$$

Examples

1. For $P = -\partial_x^2 + x\partial_y$, $\qquad \mathrm{HN}(\mathcal{F})_p = \mathfrak{t}^*\mathcal{F}_p$ for all $p \in M$.
2. For $P = -\partial_x^2 + \frac{x^2}{2}\partial_y$,
$$\mathrm{HN}(\mathcal{F})_p = \begin{cases} \mathbb{R}^{2*} = \mathfrak{t}^*\mathcal{F}_p, & x \neq 0, \\ \{(\alpha,\beta,\gamma,\delta) \mid \beta\delta = \frac{1}{2}\gamma^2\}, & x = 0. \end{cases}$$

Principal cosymbols

Definition (Helffer-Nourrigat cone)

$$\mathrm{HN}(\mathcal{F}) := \overline{\mathfrak{t}^*\mathcal{F}|_{t>0}} \cap \mathfrak{t}^*\mathcal{F}|_{t=0}$$

Examples

1. For $P = -\partial_x^2 + x\partial_y$, $\quad \mathrm{HN}(\mathcal{F})_p = \mathfrak{t}^*\mathcal{F}_p$ for all $p \in M$.
2. For $P = -\partial_x^2 + \frac{x^2}{2}\partial_y$,
$$\mathrm{HN}(\mathcal{F})_p = \begin{cases} \mathbb{R}^{2*} = \mathfrak{t}^*\mathcal{F}_p, & x \neq 0, \\ \{(\alpha,\beta,\gamma,\delta) \mid \beta\delta = \frac{1}{2}\gamma^2\}, & x = 0. \end{cases}$$

Lemma (Helffer-Nourrigat)

The Helffer-Nourrigat cone $\mathrm{HN}(\mathcal{F})_p \subseteq \mathfrak{gr}\mathcal{F}_p^*$ is invariant under the coadjoint action of $\mathrm{Gr}\mathcal{F}_p$.

Principal cosymbols

Definition (Helffer-Nourrigat cone)

$\mathrm{HN}(\mathcal{F}) := \overline{\mathfrak{t}^*\mathcal{F}|_{t>0}} \cap \mathfrak{t}^*\mathcal{F}|_{t=0}$

Examples

1. For $P = -\partial_x^2 + x\partial_y$, $\quad \mathrm{HN}(\mathcal{F})_p = \mathfrak{t}^*\mathcal{F}_p$ for all $p \in M$.
2. For $P = -\partial_x^2 + \frac{x^2}{2}\partial_y$,
$$\mathrm{HN}(\mathcal{F})_p = \begin{cases} \mathbb{R}^{2*} = \mathfrak{t}^*\mathcal{F}_p, & x \neq 0, \\ \{(\alpha,\beta,\gamma,\delta) \mid \beta\delta = \frac{1}{2}\gamma^2\}, & x = 0. \end{cases}$$

Lemma (Helffer-Nourrigat)

The Helffer-Nourrigat cone $\mathrm{HN}(\mathcal{F})_p \subseteq \mathfrak{gr}\mathcal{F}_p^*$ is invariant under the coadjoint action of $\mathrm{Gr}\mathcal{F}_p$.

Kirillov $\Rightarrow$ $\mathrm{HN}(\mathcal{F})_p$ corresponds to a family of unitary irreps of $\mathrm{Gr}\mathcal{F}_p$.

Parametrices

Theorem (Androulidakis-Mohsen-Y.)

- $P \in DO(M)$ — *polynomial in vector fields of* $\mathcal{F}$ *with total order* $\leqslant m$,
- $k \in \mathcal{E}'_r(M \times M)$ *its Schwartz kernel,*
- $\underline{k} \in \mathcal{E}'_r(\mathbb{T}\mathcal{F})$ *any extension to an ess. homog. distribution of degree* m.

Parametrices

Theorem (Androulidakis-Mohsen-Y.)

- $P \in DO(M)$ — *polynomial in vector fields of* $\mathcal{F}$ *with total order* $\leqslant m$,
- $k \in \mathcal{E}'_r(M \times M)$ *its Schwartz kernel,*
- $\underline{k} \in \mathcal{E}'_r(\mathbb{T}\mathcal{F})$ *any extension to an ess. homog. distribution of degree m.*

Then:

1. *For* $\pi \in \mathrm{HN}(\mathcal{F})_p$, *the op.* $\pi(\underline{k}|_0)$ *depends only on P (and not on* $\underline{k}$*).*

Parametrices

Theorem (Androulidakis-Mohsen-Y.)

- $P \in DO(M)$ — *polynomial in vector fields of* $\mathcal{F}$ *with total order* $\leqslant m$,
- $k \in \mathcal{E}'_r(M \times M)$ *its Schwartz kernel,*
- $\underline{k} \in \mathcal{E}'_r(\mathbb{T}\mathcal{F})$ *any extension to an ess. homog. distribution of degree* m.

Then:

1. *For* $\pi \in \mathrm{HN}(\mathcal{F})_p$, *the op.* $\pi(\underline{k}|_0)$ *depends only on* P *(and not on* $\underline{k}$*).*
2. *If* $\pi(\underline{k}|_0)$ *is left-invertible* $\forall \pi \in \mathrm{HN}(\mathcal{F})$, *then* $\exists \underline{l} \in \mathcal{E}'_r(\mathbb{T}\mathcal{F})$ *ess. homog. of degree* $-m$ *s.t.* $\underline{l} * \underline{k} - 1 = t\underline{u}$ *with* $\underline{u}$ *ess. homog. of degree* -1.

Parametrices

Theorem (Androulidakis-Mohsen-Y.)

- *$P \in DO(M)$ — polynomial in vector fields of $\mathcal{F}$ with total order $\leqslant m$,*
- *$k \in \mathcal{E}'_r(M \times M)$ its Schwartz kernel,*
- *$\underline{k} \in \mathcal{E}'_r(\mathbb{T}\mathcal{F})$ any extension to an ess. homog. distribution of degree m.*

Then:

1. *For $\pi \in \mathrm{HN}(\mathcal{F})_p$, the op. $\pi(\underline{k}|_0)$ depends only on P (and not on $\underline{k}$).*
2. *If $\pi(\underline{k}|_0)$ is left-invertible $\forall \pi \in \mathrm{HN}(\mathcal{F})$, then $\exists \underline{l} \in \mathcal{E}'_r(\mathbb{T}\mathcal{F})$ ess. homog. of degree $-m$ s.t. $\underline{l} * \underline{k} - 1 = t\underline{u}$ with $\underline{u}$ ess. homog. of degree -1.*
3. *Consequently, such P has a parametrix in $\Psi^\bullet(\mathcal{F})$, $\Rightarrow$ hypoelliptic.*

Parametrices

Theorem (Androulidakis-Mohsen-Y.)

- *$P \in DO(M)$ — polynomial in vector fields of $\mathcal{F}$ with total order $\leqslant m$,*
- *$k \in \mathcal{E}'_r(M \times M)$ its Schwartz kernel,*
- *$\underline{k} \in \mathcal{E}'_r(\mathbb{T}\mathcal{F})$ any extension to an ess. homog. distribution of degree m.*

Then:

1. *For $\pi \in \mathrm{HN}(\mathcal{F})_p$, the op. $\pi(\underline{k}|_0)$ depends only on P (and not on $\underline{k}$).*
2. *If $\pi(\underline{k}|_0)$ is left-invertible $\forall \pi \in \mathrm{HN}(\mathcal{F})$, then $\exists \underline{l} \in \mathcal{E}'_r(\mathbb{T}\mathcal{F})$ ess. homog. of degree $-m$ s.t. $\underline{l} * \underline{k} - 1 = t\underline{u}$ with $\underline{u}$ ess. homog. of degree -1.*
3. *Consequently, such P has a parametrix in $\Psi^\bullet(\mathcal{F})$, $\Rightarrow$ hypoelliptic.*

The statement suggests that the appropriate groupoid is **not** in fact $\mathbb{T}\mathcal{F}$, but a new groupoid whose rep theory at $t = 0$ is only $\mathrm{HN}(\mathcal{F})$.

Parametrices

Theorem (Androulidakis-Mohsen-Y.)

- *$P \in DO(M)$ — polynomial in vector fields of $\mathcal{F}$ with total order $\leqslant m$,*
- *$k \in \mathcal{E}'_r(M \times M)$ its Schwartz kernel,*
- *$\underline{k} \in \mathcal{E}'_r(\mathbb{T}\mathcal{F})$ any extension to an ess. homog. distribution of degree m.*

Then:

1. *For $\pi \in \mathrm{HN}(\mathcal{F})_p$, the op. $\pi(\underline{k}|_0)$ depends only on P (and not on $\underline{k}$).*
2. *If $\pi(\underline{k}|_0)$ is left-invertible $\forall \pi \in \mathrm{HN}(\mathcal{F})$, then $\exists \underline{l} \in \mathcal{E}'_r(\mathbb{T}\mathcal{F})$ ess. homog. of degree $-m$ s.t. $\underline{l} * \underline{k} - 1 = t\underline{u}$ with $\underline{u}$ ess. homog. of degree -1.*
3. *Consequently, such P has a parametrix in $\Psi^\bullet(\mathcal{F})$, $\Rightarrow$ hypoelliptic.*

The statement suggests that the appropriate groupoid is **not** in fact $\mathbb{T}\mathcal{F}$, but a new groupoid whose rep theory at $t = 0$ is only $\mathrm{HN}(\mathcal{F})$. We need:

$$\mathrm{Moh}(\mathfrak{a}\mathcal{F}) = \text{Mohsen's blow-up groupoid.}$$

Thank you