Resonances and residue operators for pseudo-Riemannian hyperbolic spaces

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Introduction

This is joint work with Jan Frahm (Aarhus University):
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- Laplace-Beltrami operator: key object in Riemannian geometry: its spectrum encodes some fine properties of the geometry of the manifold and the dynamics of the geodesic flow.

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- Laplace-Beltrami operator: key object in Riemannian geometry: its spectrum encodes some fine properties of the geometry of the manifold and the dynamics of the geodesic flow.
- problems in spectral geometry: tool from representation theory of reductive groups:
  trace formulas, Selberg trace formula, Plancherel formula, Inversion formula
- study: the resolvent, the spectrum of Laplacians, residue representations

Resonances

- Riemannian manifold X, Laplace–Beltrami operator $-\Delta$
- Resonances: the poles of a possible meromorphic extension of its resolvent

$$R(z) = (\Delta - z\,\mathrm{Id})^{-1} \qquad (z \in \mathbb{C} \setminus \sigma(\Delta))$$

- Their position in the complex plane (or a certain Riemann surface) carries refined information about the long-term behavior of the corresponding dynamical system
  $\rightsquigarrow$ they are important spectral invariants of X

Related work

- For asymptotically hyperbolic manifolds by Mazzeo–Melrose, Guillopé–Zworski and Guillarmou
- For Riemannian symmetric spaces by Miatello–Wallach, Mazzeo–Vasy, Strohmeier, Hilgert–Pasquale, and Hilgert–Pasquale–Przebinda
- In the pseudo-Riemannian setting: much less seems to be known: see e.g. Bachelot– Motet-Bachelot, Sá Barreto–Zworski: Lorentzian manifolds modelling black holes

Pseudo-Riemannian manifolds

- Pseudo-Riemannian manifolds also come with a Laplace–Beltrami operator $-\Box$.
- $\Box$ is no longer elliptic and positive, it still makes sense to consider its resolvent
- study its meromorphic continuation

In particular:

- X be a hyperbolic space: a reductive symmetric space of the form

$$X = G/H = \mathrm{U}(p,q;\mathbb{F})/(\,\mathrm{U}(1;\mathbb{F}) \times \mathrm{U}(p-1,q;\mathbb{F}))$$

$$(\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O})$$

- X carries a unique (up to scalar multiples) G-invariant pseudo-Riemannian metric of signature $(dq, d(p-1))$, where $d = \dim_{\mathbb{R}} \mathbb{F} \in \{1,2,4,8\}$
- the Laplace–Beltrami operator $-\Box$ is self-adjoint on $L^2(X)$ (with respect to the Riemannian measure)
- the resolvent is $R(z) = (\Box - z\,\mathrm{Id})^{-1} \qquad (z \in \mathbb{C} \setminus \sigma(\Box))$

- For $q = 0$, the space X is a compact Riemannian manifold, so $\sigma(\Box)$ is discrete and consists of eigenvalues, hence the resolvent $R(z)$ is already meromorphic on $\mathbb{C}$.
- If $p = 1$, then X is a non-compact Riemannian symmetric space and $\Box$ has purely continuous spectrum.
- In this case, the meromorphic extension of $R(z)$ to a Riemann surface was studied in detail by Miatello–Will, Mazzeo–Vasy and Hilgert–Pasquale.
- we assume $p \geq 2$ and $q \geq 1 \rightsquigarrow X$ is non-compact and non-Riemannian,
- put $\rho = \frac{d(p+q)-2}{2}$.

Pseudo-Riemannian hyperbolic spaces

- $n = p + q$
- consider on $\mathbb{F}^n$ the standard sesquilinear form of signature (dp, dq) given by

  $$[x, y] = \overline{y_1}x_1 + \cdots + \overline{y_p}x_p - \overline{y_{p+1}}x_{p+1} - \cdots - \overline{y_{p+q}}x_{p+q},$$

  $(x, y \in \mathbb{F}^n)$
- on the open subset $\{y \in \mathbb{F}^n : [y, y] > 0\}$, the pseudo-Riemannian metric

  $$ds^2 = -\frac{[dy, dy]}{[y, y]}$$

  is invariant under dilations $y \mapsto y\lambda$, $\lambda \in \mathbb{F}^\times = \mathbb{F} \setminus \{0\}$

Pseudo-Riemannian hyperbolic spaces

- hence it induces a pseudo-Riemannian metric g of signature $(dq, d(p-1))$ on the corresponding open subset
$$X = \{[y] \in P^{n-1}(\mathbb{F}) : [y, y] > 0\}$$
in the projective space $P^{n-1}(\mathbb{F}) = (\mathbb{F}^n \setminus \{0\})/\mathbb{F}^\times$

- This constructs a family of pseudo-Riemannian manifolds (X, g) called *hyperbolic spaces*.

- The group $G = \mathrm{U}(p, q; \mathbb{F})$ (i.e. $\mathrm{O}(p, q)$ for $\mathbb{F} = \mathbb{R}$, $\mathrm{U}(p, q)$ for $\mathbb{F} = \mathbb{C}$ and $\mathrm{Sp}(p, q)$ for $\mathbb{F} = \mathbb{H}$) preserves the form $[\cdot, \cdot]$ and hence it acts on X

- G acts transitively on X, so we can identify $X \simeq G/H$, where $H = \mathrm{U}(1; \mathbb{F}) \times \mathrm{U}(p-1, q; \mathbb{F})$ is the stabilizer of the base point $x_0 = [(1, 0, \ldots, 0)]$.

- X becomes a semisimple symmetric space

The Poisson kernel

- G: $n \times n$ matrices with entries in $\mathbb{F}$,
- choose the maximal compact subgroup $K = \mathrm{U}(p;\mathbb{F}) \times \mathrm{U}(q;\mathbb{F})$ of G
- Let

$$M = \left\{ \begin{pmatrix} a & & \\ & B & \\ & & a \end{pmatrix} : a \in \mathrm{U}(1,\mathbb{F}), B \in \mathrm{U}(p-1,q-1;\mathbb{F}) \right\},$$

$$N = \exp\left\{ \begin{pmatrix} w & z^* & -w \\ z & \mathbf{0}_{n-2} & -z \\ w & z^* & -w \end{pmatrix} : z \in \mathbb{F}^{n-2}, w \in \mathrm{Im}\,\mathbb{F} \right\},$$

with $z^* = -\overline{z}^\top \mathbf{1}_{p-1,q-1}$ and $\mathrm{Im}\,\mathbb{F} = \{w \in \mathbb{F} : \mathrm{Re}(w) = 0\}$.

- write $\Xi \simeq G/MN$

The Poisson kernel

- *Poisson kernel* $P : X \times \Xi \to \mathbb{C}$

$$P(x,\xi) = \frac{\big|[y,\eta]\big|}{\sqrt{[y,y]}},$$

where $x = [y] \in X$ and $\xi = [\eta] \in \Xi$.

- For fixed $\xi \in \Xi$, its complex powers (as functions of $x \in X$ where $P(x,\xi) \neq 0$) are eigenfunctions of the Laplace–Beltrami operator $-\Box$ on X

$$\Box P(x,\xi)^{s-\rho} = -(s^2 - \rho^2) P(x,\xi)^{s-\rho}, \qquad (s \in \mathbb{C}),$$

where $\rho = \frac{d(p+q)}{2} - 1$.

The Fourier transform

For $f \in C_c^\infty(X)$ the following integral converges whenever $\mathrm{Re}(s) > \rho - d$:

$$\mathcal{F}_s f(\xi) = \frac{1}{\Gamma(\frac{s-\rho+d}{2})} \int_X P(x,\xi)^{s-\rho} f(x)\, dx.$$

Then,

$$\mathcal{F}_s(\Box f) = -(s^2 - \rho^2)\mathcal{F}_s f.$$

The Poisson transform

Poisson transform $\mathcal{P}_s g$ of a function $g \in \mathcal{E}_s(\Xi)$:

$$\mathcal{P}_s g(x) = \frac{1}{\Gamma(\frac{-s-\rho+d}{2})} \int_B P(x,b)^{-s-\rho} g(b)\, db.$$

Then $\mathcal{P}_s g \in C^\infty(X)$ is an eigenfunction of $\square$:

$$\square(\mathcal{P}_s g) = -(s^2 - \rho^2)\mathcal{P}_s g$$

and $\mathcal{P}_s g$ depends holomorphically on $s \in \mathbb{C}$.

Spherical distributions

We define a distribution φ_s on X by

$$\langle \varphi_s, f \rangle = \mathcal{P}_s(\mathcal{F}_s f)(x_0) \qquad (f \in C_c^\infty(X)).$$

Then

$$\mathcal{P}_s(\mathcal{F}_s f) = f * \varphi_s,$$

the convolution of f with φ_s. The distributions φ_s are *spherical*, i.e. they are H-invariant and eigendistributions of the Laplacian

$$\Box \varphi_s = -(s^2 - \rho^2)\varphi_s.$$

The inversion formula

Let

$$c(s)=\frac{\Gamma(dp/2)\Gamma(dq/2)}{\sqrt{\pi}}\frac{2^{\rho-s}\Gamma(s)}{\Gamma((s+\rho)/2)\Gamma((s+dp-\rho/2))\Gamma((s+dq-\rho/2))}.$$

Theorem (J. Faraut, Distributions sphériques sur les espaces hyperboliques, 1979)

For $f\in C_c^\infty(X)$ the following inversion formula holds:
For $\mathbb{F}=\mathbb{R}$ and q odd:

$$f(x)=\frac{1}{4\pi}\int_{\mathbb{R}}(f*\varphi_{i\nu})(x)\frac{d\nu}{|c(i\nu)|^2}$$
$$+\sum_{\rho+2k+1>0}(f*\varphi_{\rho+2k+1})(x)\operatorname*{Res}_{s=\rho+2k+1}\frac{1}{c(s)c(-s)}.$$

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For $\mathbb{F}=\mathbb{R}$ and q even, or $\mathbb{F}=\mathbb{C},\mathbb{H}$:

$$f(x)=\frac{1}{4\pi}\int_{\mathbb{R}}(f*\varphi_{i\nu})(x)\frac{d\nu}{|c(i\nu)|^2}$$
$$+\sum_{0<\rho+2k<\rho}(f*\varphi_{\rho+2k})(x)\operatorname*{Res}_{s=\rho+2r}\frac{1}{c(s)c(-s)}$$
$$+\sum_{\rho+2k\geq\rho}(f*\theta_k)(x)c_{-2}\left[\frac{1}{c(s)c(-s)},\rho+2k\right].$$

Here, $c_{-2}[h(s),s_0]$ denotes the coefficient of $(s-s_0)^{-2}$ in the Laurent expansion of a meromorphic function $h(s)$ around s_0.

Meromorphic continuation

$$\widetilde{R}(\zeta) = (\Box - \rho^2 - \zeta^2)^{-1} : C_c^\infty(X) \to \mathcal{D}'(X).$$

Applying the resolvent $\widetilde{R}(\zeta)$ to the inversion formula yields

$$\widetilde{R}(\zeta)f(x) = I(\zeta)f(x) - D(\zeta)f(x),$$

where

$$I(\zeta)f(x) = \frac{1}{4\pi}\int_{\mathbb{R}} (\nu^2 - \zeta^2)^{-1}(f * \varphi_{i\nu})(x)\frac{d\nu}{|c(i\nu)|^2}$$

is the contribution of the continuous spectrum and

$$D(\zeta) = \sum_{s\in D_1} \frac{(f * \varphi_s)(x)}{\zeta^2 + s^2} \operatorname*{Res}_{\sigma=s} \frac{1}{c(\sigma)c(-\sigma)} + \sum_{\rho+2k\in D_2} \frac{(f * \theta_k)(x)}{\zeta^2 + (\rho+2k)^2} c_{-2}\left[\frac{1}{c(\sigma)c(-\sigma)}, \rho + 2k\right]$$

While $D(\zeta)$ clearly is meromorphic in $\zeta \in \mathbb{C}$, the integral expression only shows that $I(\zeta)$ is holomorphic in the upper half plane $\{\operatorname{Im}\zeta > 0\}$ and we have to shift the contour to extend it to all $\zeta \in \mathbb{C}$.

Then,

$$I(\zeta)f(x) = \frac{1}{4\pi}\int_{\mathbb{R}-iN}\frac{1}{\nu^2-\zeta^2}(f*\varphi_{i\nu})(x)\frac{d\nu}{c(i\nu)c(-i\nu)}$$
$$+\frac{i}{4\zeta}\frac{(f*\varphi_{-i\zeta})(x)}{c(i\zeta)c(-i\zeta)}+\frac{1}{2}\sum_{s\in E_N}\operatorname*{Res}_{\sigma=s}\frac{1}{\zeta^2+\sigma^2}\frac{(f*\varphi_\sigma)(x)}{c(\sigma)c(-\sigma)}.$$

Theorem (Frahm-S., 2023)

The resolvent $\widetilde{R}(\zeta) : C_c^\infty(X) \to \mathcal{D}'(X)$*, initially defined for* $\operatorname{Im}\zeta > 0$ *with* $\zeta^2 + \rho^2 \notin \sigma_p(\Box)$*, has a meromorphic extension to* $\zeta \in \mathbb{C}$*.*

Theorem (Frahm-S., 2023)

The poles of $\widetilde{R}(\zeta)$ are:

1. *In the upper half plane $\{\operatorname{Im}\zeta > 0\}$ there are single poles at $\zeta = is$, $\rho^2 - s^2 \in \sigma_p(\Box)$.*
2. *On the real line $\{\operatorname{Im}\zeta = 0\}$ the only possible pole is $\zeta = 0$, and this is indeed a single pole if and only if either $\mathbb{F} = \mathbb{R}$ and $p - q \in 4\mathbb{Z} + 2$ or $\mathbb{F} = \mathbb{C}$ and $p - q \in 2\mathbb{Z}$.*
3. *In the lower half plane $\{\operatorname{Im}\zeta < 0\}$ the poles are:*
   - 3.1 *For $\mathbb{F} = \mathbb{R}$ with p and q even or $\mathbb{F} = \mathbb{C}, \mathbb{H}, \mathbb{O}$ there are single poles at $-is$, $s \in \rho + 2\mathbb{N}$.*
   - 3.2 *For $\mathbb{F} = \mathbb{R}$ with p odd and q even there are single poles at $-is$, where either $s \in \rho + 2\mathbb{N}$ or $s \in (\rho + 2\mathbb{Z} + 1) \cap \mathbb{R}_+$.*
   - 3.3 *For $\mathbb{F} = \mathbb{R}$ with p and q odd there are no poles in the lower half plane.*
   - 3.4 *For $\mathbb{F} = \mathbb{R}$ with p even and q odd there are single poles at $-is$, $0 < s < \rho$, $s \in \rho + 2\mathbb{Z}$, and there are poles of order two at $-is$, $s \in \rho + 2\mathbb{N}$.*

Residue representations

- all singularities of $\widetilde{R}(\zeta)$ are on the imaginary axis $\zeta = is$, $s \in \mathbb{R}$
- the residue of $\widetilde{R}(\zeta)$ at $\zeta = is$ is a convolution operator of the form

$$C_c^\infty(X) \to \mathcal{D}'(X), \quad f \mapsto f * \varphi$$

  for a so-called *H-spherical distribution* $\varphi \in \mathcal{D}'(X)$, i.e. φ is an H-invariant eigendistribution: $\Box\varphi = -(s^2 - \rho^2)\varphi$
- Since $\Box$ is G-invariant, the resolvent $\widetilde{R}(\zeta)$ and its residues are G-equivariant, so their images form subrepresentations of $\mathcal{D}'(X)$ called *residue representations*.

Further questions

- What about p-adic groups?
- Vector -valued case? $\rightsquigarrow$ de Sitter space $dS^n = O(n,1)/O(n-1,1)$ and for $p=2$ Anti-de Sitter space $AdS^n = O(2,n-1)/O(1,n-1)$.

$\varepsilon\upsilon\chi\alpha\rho\iota\sigma\tau\acute{\omega}$!

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