Fourier integral operators on groupoids and evolution equations

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Jean-Marie Lescure
Based on work with Dominique Manchon and Stéphane Vassout

Université Paris Est Créteil, Laboratoire d'analyse et mathématiques appliquées

Introduction

The motivation to develop analytical tools on groupoids can be summarized as follows:
Many singular spaces, such as manifolds with corners with suitable structures on their faces or foliated spaces, can be desingularized by Lie groupoids.
Hence one may try to adapt known notions available on manifolds to Lie groupoids, and in return derive results on singular spaces.
In this talk: focus on the adaptation to Lie groupoids of the theory of Fourier integral operators (FIO).

1. Convolution of distributions
2. FIOs
3. Evolution equations.

Lie groupoids

$$G \overset{s,r}{\rightrightarrows} G^0 \quad ; \quad \upsilon : G^0 \hookrightarrow G \quad ;$$
$$m : G^{(2)} := G \underset{s,r}{\times} G \longrightarrow G \quad ; \quad \iota : G \longrightarrow G \quad ;$$

- s, r submersions; all maps C^∞;
- $r(\upsilon(x)) = s(\upsilon(x)) = x$;
- $(\gamma_1\gamma_2)\gamma_3 = \gamma_1(\gamma_2\gamma_3)$;
- $r(\gamma)\gamma = \gamma; \quad \gamma s(\gamma) = \gamma$;
- $r(\gamma^{-1}) = s(\gamma); \quad s(\gamma^{-1}) = r(\gamma)$;
- $r(\gamma_1\gamma_2) = r(\gamma_1); \quad s(\gamma_1\gamma_2) = s(\gamma_2)$;
- $\gamma\gamma^{-1} = r(\gamma); \quad \gamma^{-1}\gamma = s(\gamma)$.

Consequences: $\iota^{-1} = \iota$, υ is an embedding, m is a submersion.
Simplifying assumption : G^0 is compact.

Some examples

- Spaces: $X \rightrightarrows X$, Lie groups: $G \rightrightarrows \{e\}$, pairs : $X \times X \rightrightarrows X$.
- Vector bundles $E \rightrightarrows X$, fibred products: $H \underset{S}{\times} H \rightrightarrows H$.
- Action groupoids: $G \times X \rightrightarrows X$: $s(g,x) = x,\ r(g,x) = g.x, \ldots$
- The tangent groupoid of a manifold (A. Connes):

$$\mathbb{T}M = (\{0\} \times TM) \cup (0,1] \times M \times M \rightrightarrows [0,1] \times M,$$

  and more generally, any deformation to the normal cone:

$$\mathcal{D}(G,H) = G \times \mathbb{R}^* \cup N_H^G \times \{0\} \rightrightarrows \mathcal{D}(G^0, H^0)$$

- The b-groupoid (Monthubert-Pierrot, Nistor-Weinstein-Xu):

$$\mathring{M} \times \mathring{M} \cup \partial M \times \partial M \times \mathbb{R}_+^* \rightrightarrows M,$$

  and more generally, any blow-up (Debord-Skandalis):

$$[G \colon H] = (\mathcal{D}(G,H) \setminus H \times \mathbb{R}_+) / \mathbb{R}_+^* \qquad \text{blow-up manifold}$$

$$[G \colon H]_{r,s} \rightrightarrows [G^0 \colon H^0] \qquad \text{blow-up groupoid}$$

  where: $f : (X,Y) \to (X^0, Y^0)$,
  $[X \colon Y]_f := \left(\mathcal{D}(X,Y) \setminus \mathcal{D}(f)^{-1}(Y^0 \times \mathbb{R}_+)\right) / \mathbb{R}_+^*$.

Convolution

Let $C_c^\infty(G) := C^\infty(G, \Omega^{1/2})$, $\Omega^{1/2} = \Omega^{1/2}((r^* \oplus s^*)AG)$:

$$f_1 \star f_2(\gamma) = \int_{\eta \in G_{s(\gamma)}} f_1(\gamma\eta^{-1}) f_2(\eta) \; ; \; f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

$AG = \ker ds|_{G^0} \simeq T_{G^0}/TG^0 \to G^0$ is the Lie algebroid and $G_x = s^{-1}(x)$.
Question: To which distributions the convolution can be extended ?

$$\mathcal{D}'(G) := (C_c^\infty(G, \Omega_0))'$$

($\Omega_0 = \Omega^{1/2}((r^* \oplus s^*)TG^0)$, $C^\infty(G) \hookrightarrow \mathcal{D}'(G)$.)
For instance: $C_r^*(G) \star C_r^*(G) \subset C_r^*(G)$ and $C_r^*(G) \hookrightarrow \mathcal{D}'(G)$.

The cotangent groupoid

To $G \rightrightarrows G^0$ are associated a tangent groupoid $TG \rightrightarrows TG^0$ and a cotangent groupoid:

$$\Gamma = T^*G \rightrightarrows A^*G \qquad \text{(Coste-Dazord-Weinstein).}$$

Product in Γ:

$$(\gamma_1, \xi_1)(\gamma_2, \xi_2) = (\gamma_1\gamma_2, ({}^t dm)^{-1}(\xi_1, \xi_2))$$

as soon as $s(\gamma_1) = r(\gamma_2)$ and $(\xi_1, \xi_2) \in \mathrm{Ran}({}^t dm)$, that is:

$$-{}^t d(L_{\gamma_1} \circ i)(\xi_1) = {}^t dR_{\gamma_2}(\xi_2)$$

T^*G is a *symplectic* groupoid, that is, the graph of the multiplication is a lagrangian submanifold of $(T^*G, \omega) \times (T^*G, \omega) \times (T^*G, -\omega)$.

Convolution and wave front set

Let $W \subset \mathring{T}^*G := T^*G \setminus 0$ be a closed conic subset, set:

$$\mathcal{E}'_W(G) = \{u \in \mathcal{E}'(G) \; ; \; \mathrm{WF}(u) \subset W\}.$$

(L.-Manchon-Vassout)

- If $(W_1 \times W_2) \cap \ker m_\Gamma = \emptyset$, then $\star$ extends continuously to:

$$\mathcal{E}'_{W_1}(G) \times \mathcal{E}'_{W_2}(G) \longrightarrow \mathcal{E}'_W(G)$$

  where

$$W = (W_1 \cup 0) \cdot_\Gamma (W_2 \cup 0) \setminus 0 \subset \mathring{T}^*G.$$

This gives the usual rules for:

1. Pointwise product on X, use $G = X \rightrightarrows X$. Here: $\Gamma = T^*X \to X$
2. Composition of Schwartz kernels on $X \times X$, use : $G = X \times X \rightrightarrows X$, so that $\Gamma = T^*X \times T^*X \rightrightarrows T^*X$.
3. Convolution on a Lie group G, here $\Gamma \simeq \mathfrak{g}^* \rtimes G$.

$$\mathcal{E}'_a(G) := \{u \in \mathcal{E}'(G) \; ; \; \mathrm{WF}(u) \cap (\ker s_\Gamma \cup \ker r_\Gamma) = \emptyset\}$$

is an (involutive, unital) algebra for the convolution product.

G-pseudodifferential operators

Looking for a suitable notion of FIO on groupoids, let us recall a few things about G-PDO (Connes, . . .)

$\Psi^*(G)$ consists of continuous linear operators $P : C_c^\infty(G) \to C^\infty(G)$ such that there is a family $P_x : C_c^\infty(G_x) \to C^\infty(G_x)$ of pseudodifferential operators on G_x, $x \in G^0$, satisfying:

1. $P(f)|_{G_x} = P_x(f|_{G_x})$,
2. $P_y((R_\gamma)^* f) = (R_\gamma)^* P_y(f)$, $x = r(\gamma)$, $y = s(\gamma)$.

Actually, $\Psi^*(G) \simeq I(G, A^*G)$, the latter acting by left convolution on $C_c^\infty(G)$.

Note that A^*G is the unit space of Γ.

The calculus of G-FIO: composition

Having in mind the particular case of G-PDO, we set:
$\Phi^*(G)$ consists of lagrangian distributions $u \in I^*(G, \Lambda)$, for suitable lagrangian submanifolds $\Lambda \subset \mathring{T}^*G$, acting by convolution on $C_c^\infty(G)$.

Suitable means that Λ is conic, closed and contained in $T^*G \setminus (\ker r_\Gamma \cup \ker s_\Gamma)$, let us call it a G-relation, and it guaranties that:

$$u \in I(G, \Lambda), \qquad u \star \cdot\,,\ \cdot \star u \colon C_c^\infty(G) \to C^\infty(G).$$

L.-Vassout

Let Λ_1, Λ_2 be two G-relations such that $\Lambda_1 \times \Lambda_2 \cap \Gamma^{(2)}$ is clean. Then

1. $\Lambda = \Lambda_1 \cdot_\Gamma \Lambda_2$ is again a G-relation.
2. $$I_c^{m_1}(G, \Lambda_1) \star I^{m_2}(G, \Lambda_2) \subset I^{m_1+m_2+e/2-(n-2n^0)/4}(G, \Lambda).$$

   ($n = \dim(G)$, $n^0 = \dim(G^0)$ and $e = \operatorname{codim}(T(\Lambda_1 \times \Lambda_2) + T\Gamma^{(2)})$)

The calculus of G-FIO: product of symbols

Remember that:

$$I(G,\Lambda) \subset \mathcal{D}'(G,\Omega^{1/2}) \quad \text{with } \Omega^{1/2} = \Omega^{1/2}(\ker ds) \otimes \Omega^{1/2}(\ker dr).$$

Hörmander's principal symbol map reads here:

$$\sigma : I^m(G,\Lambda) \longrightarrow S^{[m+n/4]}(\Lambda, M_\Lambda \otimes \Omega_\Lambda^{1/2} \otimes \Omega^{1/2}(\ker ds_\Gamma)),$$

where M_Λ is the Maslov bundle of Λ.
Let $A_j \in I^{m_j}(G,\Lambda_j)$, $j = 1,2$ with Λ_j be as in the composition theorem.
A representative a of the principal symbol of $A_1 \star A_2$ is then given by:

$$\forall \delta \in \Lambda_1 \cdot_\Gamma \Lambda_2, \quad a(\delta) = \int_{(m_\Gamma)^{-1}(\delta)\cap\Lambda_1\times\Lambda_2} a_1(\delta_1)a_2(\delta_2).$$

The calculus of G-FIO: transposition

L.-Vassout

Let Λ be a G-relation and set $\Lambda^* := \iota_\Gamma(\Lambda)$

1. Λ^* is a G-relation.
2. If $A \in I_c^*(G, \Lambda)$ then $A^* \in I_c^*(G, \Lambda^*)$.
3. The following assertions are equivalent:
    - $r_\Gamma : \Lambda \to A^*G$ and $s_\Gamma : \Lambda \to A^*G$ are diffeomorphisms
    - $\Lambda\Lambda^\star = A^*G$ and $\Lambda^\star\Lambda = A^*G$.

   Such G-relations are called invertible.

The calculus of G-FIO: Egorov theorem, continuity

Since A^*G is the unit G-relation, it is obvious that $I(G,\Lambda)$ is a module over $\Psi_c^*(G)$. Also:

L-Vassout

Let Λ be an invertible G-relation. Then:

1. $$I_c^*(G,\Lambda) \star \Psi^*(G) \star I_c^*(G,\Lambda^*) \subset \Psi^*(G)$$ Egorov's theorem.

2. $$I^{(n-2n^0)/4}(G,\Lambda) \subset \mathcal{M}(C^*(G)),$$ $$I^{<(n-2n^0)/4}(G,\Lambda) \subset C^*(G).$$

The calculus of G-FIO: family of operators

Any $P \in I_c(G, \Lambda)$ gives an equivariant family of operators $P_x \in \mathcal{L}(C^\infty(G_x), C^\infty(G_x))$, $x \in G^0$.
P_x is given by an oscillatory integrals, with possibly degenerated phases.

L-Vassout

Let Λ be a G relation such that for any orbit $O = r(s^{-1}(x))$, $x \in G^0$, the intersection $T^*_{G_O} G \cap \Lambda$ is transversal. Then Λ induces a family $\Lambda_x \subset (T^* G_x)^2$ of canonical relations and for any $P \in I^m(G, \Lambda)$, the operators in the fibers are ordinary FIO:

$$P_x \in I^{m-(n-2n^{(0)})/4}(G_x \times G_x, \Lambda_x), \quad \forall x.$$

The calculus of G-FIO: operator on the base

Any $P \in I_c(G, \Lambda)$ also gives a continuous linear operator $r_*(P) : C^\infty(G^0) \longrightarrow C^\infty(G^0)$ defined by $r_*(P)(f) = P(r^* f)|_{G^0}$. Again, $r_*(P)$ is given by oscillatory integrals, with possibly degenerated phases.

L-Vassout

Let Λ be a G-relation. If the intersection $\Lambda \cap (\ker s_\Gamma + \ker r_\Gamma)$ is transversal, then Λ induces a canonical relation $\Lambda_0 \subset (T^*M)^2$ and for any $P \in I^m(G, \Lambda)$,

$$r_* P \in I^{m-(n-2n^0)/4}(G^0 \times G^0, \Lambda_0).$$

The calculus of G-FIO

It is now very easy to write down a calculus of FIO for, for instance, manifolds with boundary or fibred boundary.
A short history:

1. Manifolds: Hörmander, Guillemin, Duistermaat,...
2. Lie groups: Nielsen-Stetkær.
3. Mfds with boundary, Melrose.
4. Foliations: Kordyukov.
5. Mfds with conical sing.: Nazaikinskii-Schulze-Sternin, Nazaikinskii-Savin-Schulze-Sternin.
6. Mfds with boundary: Battisti-Coriasco-Schrohe.

Evolution equations on groupoids

Let $P \in \Psi_c^1(G)$ be elliptic, symmetric. Then P extends to a regular self-adjoint operator on the Hilbert module $C^*(G)$ (Vassout), therefore

$t \to e^{-itP} \in M(C^*(G))$ is strongly differentiable and

$$(\frac{\partial}{\partial t} + iP)e^{-itP} = 0.$$

Let χ be the flow of the Hamiltonian of the function:

$$(T^*G \setminus \ker r_{_\Gamma}) \xrightarrow{r_\Gamma} A^*G \setminus 0 \xrightarrow{\sigma_{pr}(P)} \mathbb{R}.$$

- χ is complete and homogeneous,
- $s_\Gamma \circ \chi_t = s_\Gamma$ for all t,
- $\chi_t(\alpha\beta) = \chi_t(\alpha)\beta$.

Evolution equations on groupoids

We get a family of G-relations

$$\Lambda_t = \chi_t(A^*G \setminus 0), \quad t \in \mathbb{R}$$

giving rise to a global $(\mathbb{R} \times G)$-relation

$$\Lambda = \{(t, \tau, \delta) \in T^*(\mathbb{R} \times G) \mid \tau = -p(\delta),\ \delta \in \Lambda_t\}.$$

Theorem

There exists a C^∞ family of G-FIO $U(t) \in I^{(n^1-n^0)/4}(G, \Lambda_t)$ such that $e^{-itP} - U(t)$ is a (Vassout) regularizing operator.

The family $(U_t)_t$ gives a single distribution $U \in I^{(n^1-n^0-1)/4}(G, \Lambda)$.

Evolution equations on groupoids

Main steps in the proof:

- Guess that the correct $\mathbb{R} \times G$-relation is Λ.
- For any $U \in I^m(G, \Lambda)$, check that $\frac{\partial}{\partial t} U$, $PU \in I^{m+1}(\mathbb{R} \times G, \Lambda)$.
- Check that $\frac{\partial}{\partial t} U + iPU$ actually belongs to $I^m(\mathbb{R} \times G, \Lambda)$ and has principal symbol $\mathcal{L}_{\tau + p^0}(u) + ip^1 u$.
- The transport equation: $\begin{cases} (\frac{\partial}{\partial t} + \mathcal{L}_{p^0} + ip^1)u^0 = 0 \\ u^0(0, .) = 1 \end{cases}$ has a unique solution $u^0 \in C^\infty(\Lambda)$, homogeneous of degree 0. Choose $U^0 \in I^{(n^1 - n^0 - 1)/4}(\mathbb{R} \times G, \Lambda)$ with principal symbol u^0 and suitable support. It follows that:

  $$I - U^0(0) \in \Psi^{-1}_{G,c} \text{ and } (\frac{\partial}{\partial t} + iP)U^0 = F^1 \in I^{-1+(n^1 - n^0 - 1)/4}(\mathbb{R} \times G, \Lambda).$$

- Construct $U \sim \sum_j U^j$ that satisfies the statement.

Thank You