INTERIOR KASPAROV PRODUCTS FOR ρ-CLASSES ON RIEMANNIAN FOLIATED BUNDLES

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BRIEF RECAP AND MOTIVATIONS:

$$0 \to J \to A \to A/J \to 0 \rightsquigarrow \partial \in KK^1(A/J, J)$$

boundary map

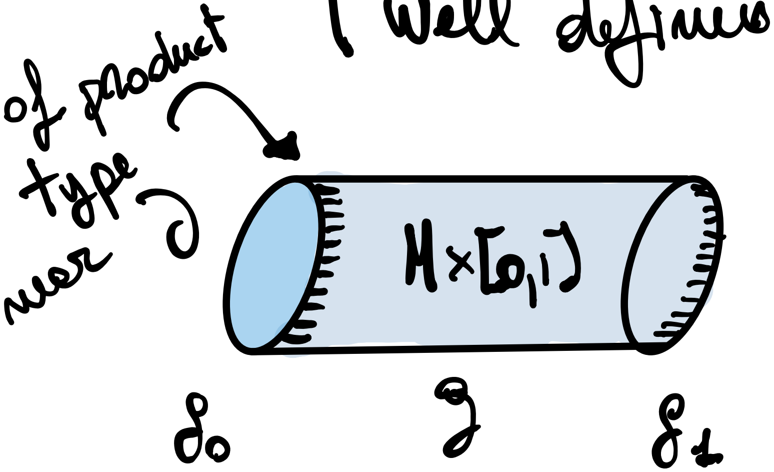
$$\begin{tikzcd}
\cdots \arrow[r] & K_*(A) \arrow[r] & K_*(A/J) \arrow[r, "\partial"] & K_{*+1}(J) \arrow[r] & \cdots \\
& \rho(\sigma) \quad \times & {[\sigma]} \arrow[u, phantom, sloped, "\in"] \arrow[l, dotted, bend right] \arrow[r] & \partial[\sigma] \arrow[u, phantom, sloped, "\in"] & \\
& \text{secondary invariant} \arrow[u] & & \text{primary invariant} \arrow[u] &
\end{tikzcd}$$

Rmk $K_*(A) =: K_*(\partial)$

Ex (M^{Spin}, g)

$$\begin{tikzcd}
{} \arrow[r] & S^{\Gamma}_{*}(\hat{M}) \arrow[r] & K_*(M) \arrow[r, "\mathrm{Ind}_\Gamma"] & K_*(C^*\Gamma) \arrow[r] & {} \\
& \varrho(g) & {[\not{D}_g]} \arrow[u, phantom, sloped, "\in"] \arrow[l, dotted, bend right, "\text{if psc}"] \arrow[r] & \mathrm{Ind}_\Gamma(\not{D}_g) \arrow[u, phantom, sloped, "\in"] &
\end{tikzcd}$$

$\uparrow$ well defined on concordance classes:



$$\Phi^+(M) := \overset{\text{psc metrics over } M}{R^+(M)} \Big/ \text{concordance}$$

(M,F) smooth spin foliation

- Connes '83: $\hat{A}(M) \neq 0 \Rightarrow \nexists\, g^F$ with p.s.c.

- Hilsum - Skandalis '87 : higher version in KK-theory

Q: Secondary version of this problem:

g_0^F, g_1^F with psc $+ \exists g^N$ s.t. $g_i^F \oplus g^N$ with psc

If $g_i^F \oplus g^N$ non concordant $\Rightarrow$ g_i^F non concordant.

LIE GROUPOIDS

$$\begin{tikzcd} G \arrow[loop left, "i"] \arrow[r, bend left=60, "t"] \arrow[r, bend left=30, "s"'] & G^{(0)} = M \arrow[l, "u"] \end{tikzcd}$$

$+$

$$\begin{tikzcd} x \arrow[r, bend left, "\gamma_1"] \arrow[rr, bend left=70, "\gamma_1 \gamma_2"] & y \arrow[r, bend left, "\gamma_2"] & z \end{tikzcd}$$

$G^{(2)} := G\,{}_s\!\times_t G \to G$ Composition law

Ex:

- $M \rightrightarrows M$ smooth manifolds
- $G \rightrightarrows *$ Lie group
- $$\begin{tikzcd} H \arrow[r, hook, "\iota"] \arrow[d, shift left] \arrow[d, shift right] & G \arrow[d, shift left] \arrow[d, shift right] \\ M & M \end{tikzcd}$$ immersion of Lie groupoids $\rightsquigarrow$ the normal bundle $N_\iota \rightrightarrows M$ is a Lie groupoid

  - $AG := N_u \rightrightarrows M$ Lie algebroid

- $\hat{M} \times_{\pi_1(M)} \tilde{M} \rightrightarrows M$ fundamental groupoid of M
- $\mathrm{Mon}(M,F) \rightrightarrows M$ monodromy groupoid of (M,F)

  ↑ homotopy classes of paths along the leaves.

GROUPOID C*-ALGEBRAS

$C_c^\infty(G, \Omega^{\frac{1}{2}})$ $*$-algebra : $f * g(\gamma) = \int_{s^{-1}(x)} f(\gamma\lambda^{-1})\, g(\lambda)$

$x = s(\gamma) \in M$

- $C^*(G)$:= closure w.r.t. all bounded representations.
- $C^*_r(G)$:= closure into the regular representation.

Exact sequences : $F \subset M$ closed union of orbits

$$0 \to C^*(G|_{M\setminus F}) \to C^*(G) \to C^*(G|_F) \to 0$$

$r^{-1}(M\setminus F) \cap s^{-1}(M\setminus F)$

$r^{-1}(F) \cap s^{-1}(F)$

KK-THEORY

A, B $\quad \Gamma$ - algebras

$(\mathcal{E}, F)$

$\mathbb{Z}_2$-graded A-B bimodule

bdd in $\mathbb{B}(\mathcal{E})$

- is a Kasparov bimodule $(\in \mathbb{E}^\Gamma(A,B))$

if $[e,F]$, $e(F^2-1)$, $e(F-F^*)$, $a(g(F)-F)$ are in $\mathbb{K}(\mathcal{E})$ $\forall a \in A$ and $g \in \Gamma$

- is degenerated $(\in \mathbb{D}^\Gamma(A,B))$

$[e,F]$, $e(F^2-1)$, $e(F-F^*)$, $a(g(F)-F) = 0$ $\forall a, g$

Ex: D invertible $\rightsquigarrow F = \frac{D}{|D|}$ gives a degenerate bimodule

$$\hookrightarrow (\mathcal{E}\otimes C_0[0,1), F\otimes id) \in \mathbb{E}^\Gamma(A, B\otimes C[0,1])$$

$$\boxed{KK^\Gamma(A,B) = \mathbb{E}^\Gamma(A,B) / \text{homotopy} + \mathbb{D}^\Gamma(A,B)}$$

Elements in $KK^\Gamma(A,B)$ can be considered as "generalized morphisms" between $K_*^\Gamma(A)$ and $K_*^\Gamma(B)$.

$$\rotatebox[origin=c]{-90}{$\rightsquigarrow$}$$

Kasparov product "=" composition of generalized morphisms.

$$KK^\Gamma(A,B) \times KK^\Gamma(B,C) \longrightarrow KK^\Gamma(A,C)$$

Rmk (after Cuntz) elements in $KK(A,B)$ can be always written as $[\varphi]^{-1} \underset{A'}{\otimes} [\phi]$ where $\varphi : A' \to A$ and $\phi : A' \to B$ are $*$-homomorphisms.

DEFORMATION TO THE NORMAL CONE

$\iota : H \hookrightarrow G$ immersion of Lie groupoids over M

DEF $DNC(\iota) := N_\iota \times \{0\} \sqcup G \times (0,1] \rightrightarrows M \times [0,1]$

Ex: $G^{[0,1]}_{ad} := DNC(u) = AG \times \{0\} \sqcup G \times (0,1]$

$[ev_0] : K_*(C^*(G^{[0,1]}_{ad})) \to K_*(C^*(AG))$ is

invertible $\Rightarrow [ev_0]^{-1} \in KK(C^*(AG), C^*(G^{[0,1]}_{ad}))$

Th $[ev_0]^{-1} \otimes [ev_1] \in KK(C^*(TM), C^*(M \times M))$

is the Atiyah-Singer index map!

In general $\overset{\text{suspension}}{S} \otimes [ev_0]^{-1} \otimes [ev_1]$ is the boundary map of

$$0 \to C^*(G) \otimes C_0(0,1) \to C^*(G_{ad}^{[0,1)}) \to C^*(AG) \to 0$$

$\cong$ mapping cone of ev_1

Rmk $\varphi : A \to B$ map C_φ mapping cone C^*-algebra

$$\mathbb{E}(\mathbb{C}, C_\varphi) \ni \left((\mathcal{E}_A, F_A), (\mathcal{E}_B^t, F_B^t) \right) \text{ s.t.}$$

Cuntz Skandalis

i) $(\mathcal{E}_B^0, F_B^0) = (\mathcal{E}_A \otimes_\varphi B, F_A \otimes 1)$

ii) $(\mathcal{E}_B^1, F_B^1) \in \mathbb{D}(A, B)$

SECONDARY INVARIANTS IN GROUPOID K-THEORY

$G \rightrightarrows M$ s.t. AG spin + g^G metric of AG

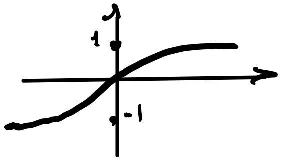
$\mathcal{E}_G := \overline{C^\infty(G, r^*\not{S}_G)}^{\|\cdot\|_{C^*(G)}}$ $\qquad \not{D}_G : \xi \mapsto \sum_\alpha c(e_\alpha)\nabla_{e_\alpha}\xi$

if $\boxed{scal(g^G) > 0}$

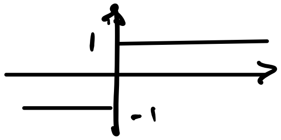
$$\begin{tikzcd}
 & \left(\mathcal{E}_{G^{[0,1]}_{ad}}, \varphi(\not{D}_{G^{[0,1]}_{ad}})\right) \arrow[dl, "ev_0"'] \arrow[dr, "ev_1"] & \\
\varphi(\hat{\sigma}(\not{D}_G)) & & \varphi(\not{D}_G)
\end{tikzcd}$$

, $\left(\mathcal{E}_{G^{[0,1]}}, \varphi_t(\not{D}_G)\right)$

$\varphi_0 = \varphi$



$\varphi_1 = sgn$



$\rightsquigarrow \boxed{\varrho(g^G) \in KK(\mathbb{C}, C^*(G^{[0,1]}_{ad}))}$ given by concatenation

THOM CLASS à la Higson-Skandalis

① Bott class $\beta_n := \left[C_0(\mathbb{R}^n, S^n) , x \mapsto \frac{cl(x)}{\sqrt{1+\|x\|^2}} \right] \in KK_n^{Spin(n)}(\mathbb{C}, C_0(\mathbb{R}^n))$

② Descent map: $j^\Gamma : KK^\Gamma(A,B) \longrightarrow KK(A \rtimes \Gamma, B \rtimes \Gamma)$

$$(\mathcal{E}, F) \longmapsto (\mathcal{E} \rtimes \Gamma, F^\rtimes)$$

③ Morita equivalence: $\Gamma \curvearrowright A$ "proper and free"

$\Rightarrow \exists\, M_\Gamma^A \in KK(A \rtimes \Gamma, A^\Gamma)$ imprimitivity bimodule.

$\iota : H \hookrightarrow G$ such that N_ι is Spin

$\Rightarrow \exists$ cocycle $\lambda : H \to Spin(n)$

$\Rightarrow$ H-equivariant principal Spin(n)-bundle P_λ over M

$$\text{s.t. } C^*(N_\iota) \cong \left(C^*(\mathbb{R}^n) \otimes C^*(P_\lambda \rtimes H) \right)^{Spin(n)}$$

Bott

Morita eq.

$$\underline{\text{Def}} \quad B(\iota) := M_{Spin} \otimes j^{Spin}\left(id_{C^*(P_\lambda \rtimes H)} \otimes \beta_n \right) \otimes M_{Spin}^{-1}$$

Morita eq.

$$\text{in} \quad KK_M\left(C^*(H), C^*(N_\iota) \right)$$

$\boxed{\text{Ex}}$ $TM \to M$ spin $\Rightarrow$ $pt^*\left(B_M(n) \right) = [\hat{\sigma}(\not{D}_M)] \in KK_M(\mathbb{C}, C^*(TM))$

$pt : M \to *$

LOWER SHRIEK MAPS

$\pi: M \to B$ Riem. submersion $\rightsquigarrow$ $g^M_\varepsilon = g^{M/B} \oplus \varepsilon^{-1} \pi^* g_B$

$d\iota : \ker d\pi \hookrightarrow TM$ $\rightsquigarrow$ $\iota : M \times_B M \hookrightarrow M \times M$

Def. $\iota_! := \beta(\iota) \otimes [ev_0]^{-1} \otimes [ev_1] \in KK(C^*(M \times_B M), C^*(M \times M))$
(Hilsum-Skandalis)

Rmk: Up to Morita eq. $\iota_!$ is the class of $\not{D}_B$ in $KK(C(B), \mathbb{C})$.

Bismut-Cheeger / **Kaad-van Suijlekom**

the class of $[\not{D}^{M/B}] \otimes \iota_!$ is represented by

$$\not{D}^M_\varepsilon = \not{D}^{M/B} \otimes 1 + \varepsilon^{\frac{1}{2}} E_\varepsilon \leftsquigarrow \text{lift of } \not{D}_B + \text{0-order terms}$$

Adiebatic shriek map

$$\begin{tikzcd}
\arrow[r] & K_*(C^*(M\times_B M)\otimes C(0,1)) \arrow[r] \arrow[d, "S\iota_!"] & K_*(C^*(M\times_B M^{[0,1)}_{ad})) \arrow[r] \arrow[d, dotted, "\iota^{ad}_!"] & K_*(C^*(\ker d\pi)) \arrow[r] \arrow[d, "d\iota_!"] & {} \\
\arrow[r] & K_*(C^*(M\times M)\otimes C(0,1)) \arrow[r] & K_*(C^*(M\times M^{[0,1)}_{ad})) \arrow[r] & K_*(C^*(TM)) \arrow[r] & {}
\end{tikzcd}$$

IDEA:

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[d, "\beta"] \\
A' \arrow[r, "\varphi'"'] & B'
\end{tikzcd}$$

$\alpha \in KK(A,A')$ and $\beta \in KK(B,B')$ $\rightsquigarrow$ $(\alpha,\beta) \in KK(C_\varphi, C_{\varphi'})$ (not canonical in general)

$+$

Lemma (Skandalis)

$F, F' \in \mathbb{B}(\mathcal{E})$ s.t. $[F,F'] = \underset{\geq \lambda > -2}{P} + \underset{\text{compact}}{K}$

$$CS_t(F,F') = (1 + \cos t \cdot \sin t\, P)^{1/2} \cdot (\cos t\, F + \sin t\, F')$$

gives an operatorial homotopy of Kasparov bimodules

Corollary Any Kasparov product of an element $(\mathcal{E}_B, F_B) \in \mathbb{D}(A,B)$ and an element $(\mathcal{E}_C, F_C) \in \mathbb{E}(B,C)$ is operatorially homotopic to $\left(\mathcal{E}_B \otimes_B \mathcal{E}_C\,,\ F_B \otimes 1\right) \in \mathbb{D}(A,C)$.

Q:

$$\wp\left(g^{M/B}\right) \otimes \iota_!^{ad} = \left(\mathcal{E}_{M\times M_{ad}^{[0,1]}},\ F_{ad}^{[0,1]}\right) \overset{\swarrow \text{ concatenation}}{\star} \left(\mathcal{E}_{M\times M}{}^{[0,1]},\ CS_t\left(F^t,\ \partial_M(\not{D}^{M/B}) \otimes 1\right)\right)$$

Is homotopic to $\downarrow$ in $\mathbb{E}\left(\mathbb{C}, C^*\left(M\times M_{ad}^{[0,1]}\right)\right)$?

$$\underset{\varepsilon \text{ suitable}}{\wp\left(g_\varepsilon^{M}\right)} = \left(\mathcal{E}_{M\times M_{ad}^{[0,1]}},\ \varphi\left(\not{D}_\varepsilon^{ad}\right)\right) \star \left(\mathcal{E}_{M\times M}{}^{[0,1]},\ \varphi_t\left(\not{D}_\varepsilon^{M}\right)\right)$$

Answer : Bismut - Cheeger

if ε small enough $\Rightarrow [\not{D}^{M/B}\otimes 1, \not{D}^{M}_{\varepsilon}] > -\varepsilon C$

depending on the size of the gap in $\sigma(\not{D}^{M/B})$

i.e. on $\mathrm{scal}(g^{M/B})$

Lemma
$\Rightarrow$ $\mathrm{Sgn}(CS_t(\not{D}^{M/B}\otimes 1, \not{D}^{M}_{\varepsilon}))$ operatorial homotopy through invertible operators from $\mathrm{Sgn}(\not{D}^{M/B}\otimes 1)$ to $\mathrm{Sgn}(\not{D}^{M}_{\varepsilon})$.

Theorem (Z. 2022)

$$e(g^{M/B}) \otimes 1^{ed}_{!} = e(g^{M}_{\varepsilon})$$

in $KK^*(\mathbb{C}, C^*(\hat{M} \underset{\Gamma}{\times} \tilde{M}^{[0,1)}_{ed}))$

FINAL REMARKS

- $L_!^{od}$ is defined for general foliations (E-theory)
- The product formula holds for Riemannian foliated bundles $\left(\begin{array}{c} M \\ \downarrow \\ B \end{array}\right) \curvearrowleft \Gamma$ isometric + free, proper compact on M
- Open question for quasi-isometric or general foliations
- Under which assumption is this product injective? $\left(\mathrm{Concordance}(M, \mathcal{F}) \hookrightarrow \mathrm{Concordance}(M) \right)$
- To do: product formulas for foliated homotopy equivalences and signature operator.

Thank you !!!