Tangent space in sub-Riemannian geometry and applications

Omar Mohsen, Paris-Saclay university

Mars, 2023

Question

1. How to connect a differential operator to its representations in the Helffer-Nourrigat set?

Question

1. How to connect a differential operator to its representations in the Helffer-Nourrigat set?
2. Why the Helffer-Nourrigat set?

Question

1. How to connect a differential operator to its representations in the Helffer-Nourrigat set?
2. Why the Helffer-Nourrigat set?
3. What about a topological index formula for the analytic index of $*$-maximally hypoelliptic differential operators?

Question

1. How to connect a differential operator to its representations in the Helffer-Nourrigat set?
2. Why the Helffer-Nourrigat set?
3. What about a topological index formula for the analytic index of $*$-maximally hypoelliptic differential operators?
4. What about the heat kernel and Weyl asymptotics of the eigenvalues?

Connes's topological space

Consider the topological space

$$M \times M \times \mathbb{R}_+^\times \sqcup TM \times \{0\}.$$

Connes's topological space

Consider the topological space

$$M \times M \times \mathbb{R}_+^\times \sqcup TM \times \{0\}.$$

The topology is given by

1. The subset $M \times M \times \mathbb{R}_+^\times$ is open with its usual topology

Connes's topological space

Consider the topological space

$$M \times M \times \mathbb{R}_+^\times \sqcup TM \times \{0\}.$$

The topology is given by

1. The subset $M \times M \times \mathbb{R}_+^\times$ is open with its usual topology
2. The subset $TM \times \{0\}$ is closed with its usual topology

Connes's topological space

Consider the topological space

$$M \times M \times \mathbb{R}_+^\times \sqcup TM \times \{0\}.$$

The topology is given by

1. The subset $M \times M \times \mathbb{R}_+^\times$ is open with its usual topology
2. The subset $TM \times \{0\}$ is closed with its usual topology
3. A sequence (or a net) $(y_n, x_n, t_n) \in M \times M \times \mathbb{R}_+^\times$ converges to $(x, \xi, 0)$ if

Connes's topological space

Consider the topological space

$$M \times M \times \mathbb{R}_+^\times \sqcup TM \times \{0\}.$$

The topology is given by

1. The subset $M \times M \times \mathbb{R}_+^\times$ is open with its usual topology
2. The subset $TM \times \{0\}$ is closed with its usual topology
3. A sequence (or a net) $(y_n, x_n, t_n) \in M \times M \times \mathbb{R}_+^\times$ converges to $(x, \xi, 0)$ if
   1. $$x_n, y_n \to x \quad t_n \to 0$$

Connes's topological space

Consider the topological space

$$M \times M \times \mathbb{R}_+^\times \sqcup TM \times \{0\}.$$

The topology is given by

1. The subset $M \times M \times \mathbb{R}_+^\times$ is open with its usual topology
2. The subset $TM \times \{0\}$ is closed with its usual topology
3. A sequence (or a net) $(y_n, x_n, t_n) \in M \times M \times \mathbb{R}_+^\times$ converges to $(x, \xi, 0)$ if
   1. $$x_n, y_n \to x \quad t_n \to 0$$
   2. $$\frac{y_n - x_n}{t_n} \to \xi.$$

Connes's topological space

Consider the topological space

$$M \times M \times \mathbb{R}_+^\times \sqcup TM \times \{0\}.$$

The topology is given by

1. The subset $M \times M \times \mathbb{R}_+^\times$ is open with its usual topology
2. The subset $TM \times \{0\}$ is closed with its usual topology
3. A sequence (or a net) $(y_n, x_n, t_n) \in M \times M \times \mathbb{R}_+^\times$ converges to $(x, \xi, 0)$ if
    1. $$x_n, y_n \to x \quad t_n \to 0$$
    2. $$\frac{y_n - x_n}{t_n} \to \xi.$$

For those familiar with blowups from algebraic geometry, the above is an open subset of blowup of $M \times M \times \mathbb{R}_+$ along $\Delta_M \times \{0\}$.

Generalization to sub-Riemmanian geometry

We think of $y - x$ as the unique constant coefficient vector field whose flow at time 1 sends x to y.

Generalization to sub-Riemanian geometry

We think of $y - x$ as the unique constant coefficient vector field whose flow at time 1 sends x to y. So we have

$$y = \exp(y - x) \cdot x$$

Generalization to sub-Riemmanian geometry

Consider $X_1, \cdots, X_r$ satisfying $(C.H)_N$ for some $N \in \mathbb{N}$.

Generalization to sub-Riemmanian geometry

Consider $X_1, \cdots, X_r$ satisfying $(C.H)_N$ for some $N \in \mathbb{N}$. For $y, x \in M$ (near the diagonal), we search solutions of the equation

$$y = \exp(Z) \cdot x$$

Generalization to sub-Riemanian geometry

Consider $X_1, \cdots, X_r$ satisfying $(C.H)_N$ for some $N \in \mathbb{N}$. For $y, x \in M$ (near the diagonal), we search solutions of the equation

$$y = \exp(Z) \cdot x$$

where Z is a constant coefficient linear combination of $X_1, \cdots, X_r$ and their iterated Lie brackets up to N.

Generalization to sub-Riemanian geometry

Consider $X_1, \cdots, X_r$ satisfying $(C.H)_N$ for some $N \in \mathbb{N}$. For $y, x \in M$ (near the diagonal), we search solutions of the equation

$$y = \exp(Z) \cdot x$$

where Z is a constant coefficient linear combination of $X_1, \cdots, X_r$ and their iterated Lie brackets up to N.
Problem?

Generalization to sub-Riemanian geometry

Consider $X_1, \cdots, X_r$ satisfying $(C.H)_N$ for some $N \in \mathbb{N}$. For $y, x \in M$ (near the diagonal), we search solutions of the equation

$$y = \exp(Z) \cdot x$$

where Z is a constant coefficient linear combination of $X_1, \cdots, X_r$ and their iterated Lie brackets up to N.
Problem? There is more than one Z !

Generalization to sub-Riemanian geometry

Consider $X_1, \cdots, X_r$ satisfying $(C.H)_N$ for some $N \in \mathbb{N}$. For $y, x \in M$ (near the diagonal), we search solutions of the equation

$$y = \exp(Z) \cdot x$$

where Z is a constant coefficient linear combination of $X_1, \cdots, X_r$ and their iterated Lie brackets up to N.
Problem? There is more than one Z !

Solution: Take the limit of the space of solution = Take the limit of manifolds.

Gromov-Hausdorff convergence

Definition

We say that a sequence of pointed metric spaces (X_n, d_n, x_n) converge in the Gromov-Hausdorff distance to (X, d, x) if

Gromov-Hausdorff convergence

Definition

We say that a sequence of pointed metric spaces (X_n, d_n, x_n) converge in the Gromov-Hausdorff distance to (X, d, x) if for any $R > 0$, there exists

$$f_n : \bar{B}_{X_n}(x_n, R) \to \bar{B}_X(x, R)$$

Gromov-Hausdorff convergence

Definition

We say that a sequence of pointed metric spaces (X_n, d_n, x_n) converge in the Gromov-Hausdorff distance to (X, d, x) if for any $R > 0$, there exists

$$f_n : \bar{B}_{X_n}(x_n, R) \to \bar{B}_X(x, R)$$

such that

$$\lim_{n\to\infty} \sup_{a,b\in X_n} |d_n(a,b) - d(f_n(a), f_n(b))| = 0$$

Gromov-Hausdorff convergence

Definition

We say that a sequence of pointed metric spaces (X_n, d_n, x_n) converge in the Gromov-Hausdorff distance to (X, d, x) if for any $R > 0$, there exists

$$f_n : \bar{B}_{X_n}(x_n, R) \to \bar{B}_X(x, R)$$

such that

$$\lim_{n\to\infty} \sup_{a,b\in X_n} |d_n(a,b) - d(f_n(a), f_n(b))| = 0$$

and similarly $g_n : \bar{B}_X(x, R) \to \bar{B}_{X_n}(x_n, R)$ with similar properties.

Gromov-Hausdorff convergence

Definition

We say that a sequence of pointed metric spaces (X_n, d_n, x_n) converge in the Gromov-Hausdorff distance to (X, d, x) if for any $R > 0$, there exists

$$f_n : \bar{B}_{X_n}(x_n, R) \to \bar{B}_X(x, R)$$

such that

$$\lim_{n\to\infty} \sup_{a,b\in X_n} |d_n(a,b) - d(f_n(a), f_n(b))| = 0$$

and similarly $g_n : \bar{B}_X(x, R) \to \bar{B}_{X_n}(x_n, R)$ with similar properties.

Remark : If limit exists it is unique up to isometry

Gromov-Hausdorff convergence

Definition

We say that a sequence of pointed metric spaces (X_n, d_n, x_n) converge in the Gromov-Hausdorff distance to (X, d, x) if for any $R > 0$, there exists

$$f_n : \bar{B}_{X_n}(x_n, R) \to \bar{B}_X(x, R)$$

such that

$$\lim_{n\to\infty} \sup_{a,b\in X_n} |d_n(a,b) - d(f_n(a), f_n(b))| = 0$$

and similarly $g_n : \bar{B}_X(x, R) \to \bar{B}_{X_n}(x_n, R)$ with similar properties.

Remark : If limit exists it is unique up to isometry

Example : If d is a Riemmanian metric on M, then

$$\lim_{t\to 0^+} (M, \frac{d}{t}, x) = (T_xM, d_{costant}, x)$$

Connection to the tangent groupoid

Consider

$$M \times M \times \mathbb{R}_+^\times \sqcup TM \times \{0\} \to \mathbb{R}$$
$$(y, x, t) \mapsto \frac{d(y, x)}{t}$$
$$(x, \xi, 0) \mapsto \|\xi\|$$

It is continuous and (almost) proper.

Connection to the tangent groupoid

Consider

$$\begin{aligned} M \times M \times \mathbb{R}_+^\times \sqcup TM \times \{0\} &\to \mathbb{R} \\ (y, x, t) &\mapsto \frac{d(y, x)}{t} \\ (x, \xi, 0) &\mapsto \|\xi\| \end{aligned}$$

It is continuous and (almost) proper.

So what we put at 0 is the limit in the sense of Gromov-Hausdorff

Sub-Riemmanian metric

Take $y, x \in M$. The distance $d_{sR}(y,x) = \inf |||Z|||$ where inf is over all Z such that $y = \exp(Z) \cdot x$ and if

$$Z = \sum \alpha_i X_i + \sum \beta_{ij}[X_i, X_j] + \sum \gamma_{ijk}[[X_i, X_j], X_k] + \cdots$$

, then

$$|||Z||| = \sum |\alpha_i| + \sum |\beta_{ij}|^{\frac{1}{2}} + \sum |\gamma_{ijk}|^{1/3} + \cdots$$

What is the tangent space in sub-Riemanian geometry

What is the limit

$$\lim_{t \to 0^+} (M, \frac{d_{sR}}{t}, x)?$$

What is the tangent space in sub-Riemanian geometry

What is the limit

$$\lim_{t\to 0^+}(M, \frac{d_{sR}}{t}, x)?$$

Let G be the free nilpotent group of degree N with as many generators as X_i's.

What is the tangent space in sub-Riemanian geometry

What is the limit

$$\lim_{t \to 0^+} (M, \frac{d_{sR}}{t}, x)?$$

Let G be the free nilpotent group of degree N with as many generators as X_i's. For each $x \in M$, Bellaiche (1996) defines a connected subgroup $\mathfrak{r}_x \subseteq G$ and he proves that

$$\lim_{t \to 0^+} (M, \frac{d_{sR}}{t}, x) = (G/\mathfrak{r}_x, d_{sR}, \mathfrak{r}_x).$$

What is the tangent space in sub-Riemanian geometry

What is the limit

$$\lim_{t\to 0^+}(M, \frac{d_{sR}}{t}, x)?$$

Let G be the free nilpotent group of degree N with as many generators as X_i's. For each $x \in M$, Bellaiche (1996) defines a connected subgroup $\mathfrak{r}_x \subseteq G$ and he proves that

$$\lim_{t\to 0^+}(M, \frac{d_{sR}}{t}, x) = (G/\mathfrak{r}_x, d_{sR}, \mathfrak{r}_x).$$

Furthermore $\dim(G/\mathfrak{r}_x) = \dim(M)$.

What is the tangent space in sub-Riemanian geometry

What is the limit

$$\lim_{t\to 0^+}(M, \frac{d_{sR}}{t}, x)?$$

Let G be the free nilpotent group of degree N with as many generators as X_i's. For each $x \in M$, Bellaiche (1996) defines a connected subgroup $\mathfrak{r}_x \subseteq G$ and he proves that

$$\lim_{t\to 0^+}(M, \frac{d_{sR}}{t}, x) = (G/\mathfrak{r}_x, d_{sR}, \mathfrak{r}_x).$$

Furthermore $\dim(G/\mathfrak{r}_x) = \dim(M)$.
Is the limit uniform?

Other tangent cones

Let $x_n \to x$ and $t_n \to 0^+$. Then what is

$$\lim_{n \to +\infty} (M, \frac{d_{sR}}{t_n}, x_n)$$

Other tangent cones

Let $x_n \to x$ and $t_n \to 0^+$. Then what is

$$\lim_{n\to+\infty}(M, \frac{d_{sR}}{t_n}, x_n)$$

Answer it is always of the form $(G/V, d_{sR}, V)$ but in general $V \neq \mathfrak{r}_x$.

Example

Take ∂_x and $x^k\partial_y$ on $\mathbb{R}^2$.

Other tangent cones

Let $x_n \to x$ and $t_n \to 0^+$. Then what is

$$\lim_{n \to +\infty} (M, \frac{d_{sR}}{t_n}, x_n)$$

Answer it is always of the form $(G/V, d_{sR}, V)$ but in general $V \neq \mathfrak{r}_x$.

Example

Take ∂_x and $x^k \partial_y$ on $\mathbb{R}^2$. For each $\lambda \in \mathbb{P}^1(\mathbb{R})$, we define a connected subgroup $V_\lambda \subseteq G$ (all pairwise distinct).

Other tangent cones

Let $x_n \to x$ and $t_n \to 0^+$. Then what is

$$\lim_{n\to+\infty}(M, \frac{d_{sR}}{t_n}, x_n)$$

Answer it is always of the form $(G/V, d_{sR}, V)$ but in general $V \neq \mathfrak{r}_x$.

Example

Take ∂_x and $x^k\partial_y$ on $\mathbb{R}^2$. For each $\lambda \in \mathbb{P}^1(\mathbb{R})$, we define a connected subgroup $V_\lambda \subseteq G$ (all pairwise distinct). Then $(x_n, y_n) \to (0,0)$ and $t_n \to 0^+$.

Other tangent cones

Let $x_n \to x$ and $t_n \to 0^+$. Then what is

$$\lim_{n\to+\infty}(M, \frac{d_{sR}}{t_n}, x_n)$$

Answer it is always of the form $(G/V, d_{sR}, V)$ but in general $V \neq \mathfrak{r}_x$.

Example

Take ∂_x and $x^k\partial_y$ on $\mathbb{R}^2$. For each $\lambda \in \mathbb{P}^1(\mathbb{R})$, we define a connected subgroup $V_\lambda \subseteq G$ (all pairwise distinct). Then $(x_n, y_n) \to (0,0)$ and $t_n \to 0^+$. Consider $[x_n : t_n] \in \mathbb{P}^1(\mathbb{R})$. We show that

$$\lim_{n\to+\infty}(\mathbb{R}^2, \frac{d_{CC}}{t_n}, (x_n, y_n)) = (G/V_\lambda, d_{sR}, V_\lambda)$$

if and only if $[x_n : t_n] \to \lambda$.

Other tangent cones

In general, for each $x \in M$ and $t > 0$, we define a subspace $h_{x,t} \subseteq \mathfrak{g}$ of codimension $\dim(M)$.

Other tangent cones

In general, for each $x \in M$ and $t > 0$, we define a subspace $h_{x,t} \subseteq \mathfrak{g}$ of codimension $\dim(M)$.

Theorem (M. 2022)

Let $x_n \to x$ and $t_n \to 0$ and $h_{x_n,t_n} \to \mathfrak{v}$, then

1. *$\mathfrak{v}$ is a Lie subalgebra of $\mathfrak{g}$.*
2. $$\lim_{n \to +\infty} (M, \frac{d_{CC}}{t_n}, x_n) = (G/V, d_{sR}, V)$$

   where V is the connected subgroup integrating $\mathfrak{v}$.

Let $\mathcal{G}_x$ be the set of all V subgroups that can appear.

Connes's space in sub-Riemannian geometry

Let $\mathcal{G}_x$ be the set of all V subgroups that can appear.

$$M \times M \times \mathbb{R}_+^\times \sqcup_{x\in M, V\in\mathcal{G}_x} G/V \times \{0\}$$

Let $\mathcal{G}_x$ be the set of all V subgroups that can appear.

$$M \times M \times \mathbb{R}_+^\times \sqcup_{x \in M, V \in \mathcal{G}_x} G/V \times \{0\}$$

The topology is given by

1. The subset $M \times M \times \mathbb{R}_+^\times$ is open with its usual topology

Connes's space in sub-Riemannian geometry

Let $\mathcal{G}_x$ be the set of all V subgroups that can appear.

$$M \times M \times \mathbb{R}_+^\times \sqcup_{x\in M, V\in\mathcal{G}_x} G/V \times \{0\}$$

The topology is given by

1. The subset $M \times M \times \mathbb{R}_+^\times$ is open with its usual topology
2. $\sqcup_{x\in M, V\in\mathcal{G}_x} G/V \times \{0\}$ is a closed subset with the subquotient topology from $M \times \mathrm{Grass}(\mathfrak{g}) \times G$

Connes's space in sub-Riemannian geometry

A sequence (y_n, x_n, t_n) converges to $(x, gV, 0)$

Connes's space in sub-Riemannian geometry

A sequence (y_n, x_n, t_n) converges to $(x, gV, 0)$ if

1. $y_n, x_n \to x$ and $t_n \to 0$

Connes's space in sub-Riemannian geometry

A sequence (y_n, x_n, t_n) converges to $(x, gV, 0)$ if

1. $y_n, x_n \to x$ and $t_n \to 0$
2. $h_{x_n,t_n} \to \mathfrak{v}$ or equivalently $\lim_{n\to+\infty}(M, \frac{d_{sR}}{t_n}, x_n) = (G/V, d_{sR}, V)$.

Connes's space in sub-Riemannian geometry

A sequence (y_n, x_n, t_n) converges to $(x, gV, 0)$ if

1. $y_n, x_n \to x$ and $t_n \to 0$
2. $h_{x_n,t_n} \to \mathfrak{v}$ or equivalently $\lim_{n\to+\infty}(M, \frac{d_{sR}}{t_n}, x_n) = (G/V, d_{sR}, V)$.
3. There exists Z_n such that $y_n = \exp(Z_n) \cdot x_n$ and $\alpha_{\frac{1}{t_n}}(Z_n)$ converges to an element in gV.

Connes's space in sub-Riemannian geometry

Properties of

$$M \times M \times \mathbb{R}_+^\times \sqcup_{x \in M, V \in \mathcal{G}_x} G/V \times \{0\}$$

Connes's space in sub-Riemannian geometry

Properties of

$$M \times M \times \mathbb{R}_+^\times \sqcup_{x \in M, V \in \mathcal{G}_x} G/V \times \{0\}$$

1. It is locally compact Hausdorff second countable

Connes's space in sub-Riemannian geometry

Properties of

$$M \times M \times \mathbb{R}_+^\times \sqcup_{x\in M, V\in \mathcal{G}_x} G/V \times \{0\}$$

1. It is locally compact Hausdorff second countable (so metrizable by general topology)

Connes's space in sub-Riemannian geometry

Properties of

$$M \times M \times \mathbb{R}_+^\times \sqcup_{x \in M, V \in \mathcal{G}_x} G/V \times \{0\}$$

1. It is locally compact Hausdorff second countable (so metrizable by general topology)
2. It is not a smooth manifold

Connes's space in sub-Riemannian geometry

Properties of

$$M \times M \times \mathbb{R}_+^\times \sqcup_{x \in M, V \in \mathcal{G}_x} G/V \times \{0\}$$

1. It is locally compact Hausdorff second countable (so metrizable by general topology)
2. It is not a smooth manifold (If you think vector fields like $X = \sin(1/x)e^{-\frac{1}{x^2}}\frac{\partial}{\partial x}$ exists then it isn't a smooth manifold)

What is the Helffer-Nourrigat set?

What is the Helffer-Nourrigat set?

For each $x \in M$, let

$$\mathcal{T}_x = \sqcup_{V \in \mathcal{G}_x} \mathfrak{v}^{\perp} \subseteq \mathfrak{g}^*.$$

Notice that $\mathfrak{v}^{\perp}$ is the contangent bundle of G/V at V.

What is the Helffer-Nourrigat set?

For each $x \in M$, let

$$\mathcal{T}_x = \sqcup_{V \in \mathcal{G}_x} \mathfrak{v}^\perp \subseteq \mathfrak{g}^*.$$

Notice that $\mathfrak{v}^\perp$ is the contangent bundle of G/V at V.

Proposition

$\mathcal{T}_x$ is equal to the set Helffer and Nourrigat defined (1979)

What is the Helffer-Nourrigat set?

For each $x \in M$, let

$$\mathcal{T}_x = \sqcup_{V \in \mathcal{G}_x} \mathfrak{v}^{\perp} \subseteq \mathfrak{g}^*.$$

Notice that $\mathfrak{v}^{\perp}$ is the contangent bundle of G/V at V.

Proposition

$\mathcal{T}_x$ is equal to the set Helffer and Nourrigat defined (1979)

Proof.

Grassmanian maifolds are compact. □

What is the Helffer-Nourrigat conjecture?

What is the Helffer-Nourrigat conjecture?

Dictionary between sub-Riemannian geometry and Riemannian geometry

$$\begin{aligned} \text{maximally hypoelliptic} &= \text{elliptic} \\ \text{Helffer-Nourrigat set} &= \text{contangent space} \\ \pi(D) &= \text{classical principal} \end{aligned}$$

Where do subelliptic estimates come from?

Where do subelliptic estimates come from?

Let $x \in M$ and $V \in \mathcal{G}_x$. Consider

$$\pi_V : G \to B(L^2 G/V)$$

Where do subelliptic estimates come from?

Let $x \in M$ and $V \in \mathcal{G}_x$. Consider

$$\pi_V : G \to B(L^2 G/V)$$

Let $\mathrm{supp}(\pi_V) \subseteq \hat{G}$ be its support.

Where do subelliptic estimates come from?

Let $x \in M$ and $V \in \mathcal{G}_x$. Consider

$$\pi_V : G \to B(L^2 G/V)$$

Let $\operatorname{supp}(\pi_V) \subseteq \hat{G}$ be its support.

Proposition

$\cup_{V \in \mathcal{G}_x} \operatorname{supp}(\pi_V)$ *is the Helffer-Nourrigat set seen as a set of representations.*

Where do subelliptic estimates come from?

The space

$$M \times M \times \mathbb{R}_+^\times \sqcup_{x\in M, V\in\mathcal{G}_x} G/V \times \{0\}$$

is a topological groupoid.

Where do subelliptic estimates come from?

The space

$$M \times M \times \mathbb{R}_+^\times \sqcup_{x \in M, V \in \mathcal{G}_x} G/V \times \{0\}$$

is a topological groupoid. Its space is objects is

$$M \times \mathbb{R}_+^\times \sqcup_{x \in M, V \in \mathcal{G}_x} \{V\}$$

Proposition (M. 2021)

Let $G \rightarrow G^0$ be a topological groupoid,

Where do subelliptic estimates come from?

The space

$$M \times M \times \mathbb{R}_+^\times \sqcup_{x \in M, V \in \mathcal{G}_x} G/V \times \{0\}$$

is a topological groupoid. Its space is objects is

$$M \times \mathbb{R}_+^\times \sqcup_{x \in M, V \in \mathcal{G}_x} \{V\}$$

Proposition (M. 2021)

Let $G \to G^0$ be a topological groupoid, X a space with $\pi : G^0 \to X$ a continuous map such that $\pi \circ s = \pi \circ r$. That is

$$G = \sqcup_{x \in X} G_x$$

algebraically. Let $x_0 \in X$.

Where do subelliptic estimates come from?

The space

$$M \times M \times \mathbb{R}_+^\times \sqcup_{x \in M, V \in \mathcal{G}_x} G/V \times \{0\}$$

is a topological groupoid. Its space is objects is

$$M \times \mathbb{R}_+^\times \sqcup_{x \in M, V \in \mathcal{G}_x} \{V\}$$

Proposition (M. 2021)

Let $G \to G^0$ be a topological groupoid, X a space with $\pi : G^0 \to X$ a continuous map such that $\pi \circ s = \pi \circ r$. That is

$$G = \sqcup_{x \in X} G_x$$

algebraically. Let $x_0 \in X$. If G is amenable and

Where do subelliptic estimates come from?

The space

$$M \times M \times \mathbb{R}_+^\times \sqcup_{x \in M, V \in \mathcal{G}_x} G/V \times \{0\}$$

is a topological groupoid. Its space is objects is

$$M \times \mathbb{R}_+^\times \sqcup_{x \in M, V \in \mathcal{G}_x} \{V\}$$

Proposition (M. 2021)

Let $G \to G^0$ be a topological groupoid, X a space with $\pi : G^0 \to X$ a continuous map such that $\pi \circ s = \pi \circ r$. That is

$$G = \sqcup_{x \in X} G_x$$

algebraically. Let $x_0 \in X$. If G is amenable and $G_{x_0} \subseteq \overline{\cup_{x \neq x_0} G_x}$.

Where do subelliptic estimates come from?

The space

$$M \times M \times \mathbb{R}_+^\times \sqcup_{x\in M, V\in\mathcal{G}_x} G/V \times \{0\}$$

is a topological groupoid. Its space is objects is

$$M \times \mathbb{R}_+^\times \sqcup_{x\in M, V\in\mathcal{G}_x} \{V\}$$

Proposition (M. 2021)

Let $G \to G^0$ be a topological groupoid, X a space with $\pi : G^0 \to X$ a continuous map such that $\pi \circ s = \pi \circ r$. That is

$$G = \sqcup_{x\in X} G_x$$

algebraically. Let $x_0 \in X$. If G is amenable and $G_{x_0} \subseteq \overline{\cup_{x\neq x_0} G_x}$. Then for any $f \in C_c(G)$,

$$\limsup_{x\to x_0} \|f_{|G_x}\|_{C^*G_x} = \|f_{|G_{x_0}}\|_{C^*G_{x_0}}$$

Where do subelliptic estimates come from?

For any f continuous with compact support on our space, we have

$$\limsup_{t\to 0^+} \|f_{|M\times M\times\{t\}}\|_{B(L^2M)} = \sup_{x\in M, V\in\mathcal{G}_x} \|\pi_V(f_0)\| = \sup_{\pi} \|\pi(f_0)\|$$

where last $\sup$ is over representations in the Helffer-Nourrigat set (over all $x \in M$).

Where do subelliptic estimates come from?

For any f continuous with compact support on our space, we have

$$\limsup_{t\to 0^+} \|f_{|M\times M\times\{t\}}\|_{B(L^2M)} = \sup_{x\in M, V\in\mathcal{G}_x} \|\pi_V(f_0)\| = \sup_{\pi} \|\pi(f_0)\|$$

where last $\sup$ is over representations in the Helffer-Nourrigat set (over all $x\in M$).
Our joint work with Androulidakis and Yuncken is to transfer the identity to an identity about pseudodifferential operators.

Where do subelliptic estimates come from?

For any f continuous with compact support on our space, we have

$$\limsup_{t\to 0^+} \|f_{|M\times M\times\{t\}}\|_{B(L^2M)} = \sup_{x\in M, V\in\mathcal{G}_x} \|\pi_V(f_0)\| = \sup_{\pi} \|\pi(f_0)\|$$

where last $\sup$ is over representations in the Helffer-Nourrigat set (over all $x \in M$).
Our joint work with Androulidakis and Yuncken is to transfer the identity to an identity about pseudodifferential operators.
If P is a classical pseudodifferential operator of order 0, then

$$\|P\|_{\frac{B(L^2M)}{K(L^2M)}} = \sup_{\xi\in T^*M\setminus\{0\}} |\sigma^0(P,\xi)|.$$

Our joint work is creat a pseudodifferential calculus and then show that

$$\|P\|_{\frac{B(L^2M)}{K(L^2M)}} = \sup_{\pi\neq 1_G} \|\pi(P)\|.$$

Weights

All what we did only applies to operators like

$$X_1^2 + \cdots X_n^2$$

but not

$$X_1^2 + \cdots + X_n^2 + X_0$$

Weights

All what we did only applies to operators like

$$X_1^2 + \cdots X_n^2$$

but not

$$X_1^2 + \cdots + X_n^2 + X_0$$

Solution: Add weights. Suppose that each vector field comes with a weight which is a natural number ≥ 1

Index theory

Let $X_1, \cdots, X_n$ be vector fields satisfying Hormander's condition on a compact manifold, each equipped with a weight (natural number ≥ 1)

Index theory

Let $X_1, \cdots, X_n$ be vector fields satisfying Hormander's condition on a compact manifold, each equipped with a weight (natural number ≥ 1)

Definition

A $*$-maximally hypoelliptic differential operator is a differential operator D such that

$$\pi(D) : C^\infty(\pi) \to C^\infty(\pi)$$

is bijective for each nontrivial representation in the Helffer-Nourrigat set.

Index theory

Let $X_1, \cdots, X_n$ be vector fields satisfying Hormander's condition on a compact manifold, each equipped with a weight (natural number ≥ 1)

Definition

A $*$-maximally hypoelliptic differential operator is a differential operator D such that

$$\pi(D) : C^\infty(\pi) \to C^\infty(\pi)$$

is bijective for each nontrivial representation in the Helffer-Nourrigat set.

Corollary

If D is $*$-maximally hypoelliptic, then $D : C^\infty(M) \to C^\infty(M)$ has finite dimensional kernel and cokernel.

Index theorem

Theorem (M. 2022)

Let $X_1, \cdots, X_m$ be vector fields (with weights) satisfying Hörmander's condition, D $*$-maximally hypoelliptic on any compact manifold M.

Index theorem

Theorem (M. 2022)

Let $X_1, \cdots, X_m$ be vector fields (with weights) satisfying Hörmander's condition, D $*$-maximally hypoelliptic on any compact manifold M. Then

$$\mathrm{Ind}_a(D) = \mathrm{Ind}_{AS}(Ex(\mu(\sigma(D))))$$

Index theorem

Theorem (M. 2022)

Let $X_1, \cdots, X_m$ *be vector fields (with weights) satisfying Hörmander's condition,* D $*$*-maximally hypoelliptic on any compact manifold* M*. Then*

$$\mathrm{Ind}_a(D) = \mathrm{Ind}_{AS}(Ex(\mu(\sigma(D))))$$

Index theorem

Theorem (M. 2022)

Let $X_1, \cdots, X_m$ be vector fields (with weights) satisfying Hörmander's condition, D $*$-maximally hypoelliptic on any compact manifold M. Then

$$\mathrm{Ind}_a(D) = \mathrm{Ind}_{AS}(Ex(\mu(\sigma(D))))$$

Special cases (contact manifolds) obtained by van Erp and Baum.

Index theorem

Theorem (M. 2022)

Let $X_1, \cdots, X_m$ be vector fields (with weights) satisfying Hörmander's condition, D $*$-maximally hypoelliptic on any compact manifold M. Then

$$\mathrm{Ind}_a(D) = \mathrm{Ind}_{AS}(Ex(\mu(\sigma(D))))$$

Special cases (contact manifolds) obtained by van Erp and Baum.

- D $*$-maximally hypoelliptic implies $[\sigma(D)] \in K_0(C^*\mathcal{T})$.

Index theorem

Theorem (M. 2022)

Let $X_1, \cdots, X_m$ be vector fields (with weights) satisfying Hörmander's condition, D $*$-maximally hypoelliptic on any compact manifold M. Then

$$\mathrm{Ind}_a(D) = \mathrm{Ind}_{AS}(Ex(\mu(\sigma(D))))$$

Special cases (contact manifolds) obtained by van Erp and Baum.

- D $*$-maximally hypoelliptic implies $[\sigma(D)] \in K_0(C^*\mathcal{T})$.
- One has an isomorphism

$$\mu : K_0(C^*\mathcal{T}) \to K^0(\mathcal{T}).$$

The space $\mathcal{T} = \cup_{x \in M}\{x\} \times \mathcal{T}_x \subseteq M \times \mathfrak{g}^*$ is the union of all $\mathcal{T}_x$

Index theorem

Theorem (M. 2022)

Let $X_1, \cdots, X_m$ be vector fields (with weights) satisfying Hörmander's condition, D $*$-maximally hypoelliptic on any compact manifold M. Then

$$\mathrm{Ind}_a(D) = \mathrm{Ind}_{AS}(Ex(\mu(\sigma(D))))$$

Special cases (contact manifolds) obtained by van Erp and Baum.

- D $*$-maximally hypoelliptic implies $[\sigma(D)] \in K_0(C^*\mathcal{T})$.
- One has an isomorphism

$$\mu : K_0(C^*\mathcal{T}) \to K^0(\mathcal{T}).$$

  The space $\mathcal{T} = \cup_{x \in M}\{x\} \times \mathcal{T}_x \subseteq M \times \mathfrak{g}^*$ is the union of all $\mathcal{T}_x$
- One has Excision map $Ex : K^0(\mathcal{T}^*) \to K^0(T^*M)$

Index theorem

Theorem (M. 2022)

Let $X_1, \cdots, X_m$ be vector fields (with weights) satisfying Hörmander's condition, D $*$-maximally hypoelliptic on any compact manifold M. Then

$$\mathrm{Ind}_a(D) = \mathrm{Ind}_{AS}(Ex(\mu(\sigma(D))))$$

Special cases (contact manifolds) obtained by van Erp and Baum.

- D $*$-maximally hypoelliptic implies $[\sigma(D)] \in K_0(C^*\mathcal{T})$.
- One has an isomorphism

  $$\mu : K_0(C^*\mathcal{T}) \to K^0(\mathcal{T}).$$

  The space $\mathcal{T} = \cup_{x \in M}\{x\} \times \mathcal{T}_x \subseteq M \times \mathfrak{g}^*$ is the union of all $\mathcal{T}_x$
- One has Excision map $Ex : K^0(\mathcal{T}^*) \to K^0(T^*M)$ and $\mathrm{Ind}_{AS} : K_0(T^*M) \to \mathbb{Z}$.

Proof

We cant to connect

$$M \times M \times]0,1] \sqcup_{x \in M, V \in \mathcal{G}_x} G/V \times \{0\}$$

to

$$M \times M \times]0,1] \sqcup TM \times \{0\}$$

Proof

We cant to connect

$$M \times M \times]0,1] \sqcup_{x \in M, V \in \mathcal{G}_x} G/V \times \{0\}$$

to

$$M \times M \times]0,1] \sqcup TM \times \{0\}$$

$$\begin{array}{ccc} TM & \text{———} & M \times M \\ | & & | \\ \sqcup_{x \in M, V \in \mathcal{G}_x} \frac{\mathfrak{g}}{\mathfrak{v}} & \text{———} & \sqcup_{x \in M, V \in \mathcal{G}_x} G/V \end{array}$$

Example

Let $k \in \mathbb{N}$ be arbitrary. Consider ∂_x and $\sin(x)^k \partial_y$ on $S^1 \times S^1$.

Example

Let $k \in \mathbb{N}$ be arbitrary. Consider ∂_x and $\sin(x)^k\partial_y$ on $S^1 \times S^1$. Associate weight 1 and k respectively.

Example

Let $k \in \mathbb{N}$ be arbitrary. Consider ∂_x and $\sin(x)^k \partial_y$ on $S^1 \times S^1$. Associate weight 1 and k respectively. Consider

$$D = (-1)^k \partial_x^{2k} - \sin(x)^{2k} \partial_y^2 + ig(x,y)\partial_y$$

where $g : S^1 \times S^1 \to \mathbb{C}$ smooth.

Example

Let $k \in \mathbb{N}$ be arbitrary. Consider ∂_x and $\sin(x)^k\partial_y$ on $S^1 \times S^1$. Associate weight 1 and k respectively. Consider

$$D = (-1)^k\partial_x^{2k} - \sin(x)^{2k}\partial_y^2 + ig(x,y)\partial_y$$

where $g : S^1 \times S^1 \to \mathbb{C}$ smooth.
For any $(x,y) \in S^1 \times S^1$, $(\xi,\eta) \in \mathbb{R}^2$, the Helffer-Nourrigat set at (x,y) contains a 1-dimensional representation parametrised with (ξ,η) for which

$$\pi(D) = \xi^{2k} + \eta^2$$

Example

If $\sin(x) = 0$, then there exists two infinite dimensional representations on $L^2\mathbb{R}$ with

$$\pi_{\pm}(D) = (-1)^k \partial_z^{2k} + z^{2k} \mp g(x, y) Id_{L^2\mathbb{R}}$$

Example

If $\sin(x) = 0$, then there exists two infinite dimensional representations on $L^2\mathbb{R}$ with

$$\pi_{\pm}(D) = (-1)^k \partial_z^{2k} + z^{2k} \mp g(x,y) Id_{L^2\mathbb{R}}$$

D is $*$-maximally hypoelliptic iff

$$g(0,y), g(\pi,y) \notin \pm\mathrm{spec}((-1)^k \partial_z^{2k} + z^{2k})$$

Example

If $\sin(x) = 0$, then there exists two infinite dimensional representations on $L^2\mathbb{R}$ with

$$\pi_{\pm}(D) = (-1)^k \partial_z^{2k} + z^{2k} \mp g(x,y) Id_{L^2\mathbb{R}}$$

D is $*$-maximally hypoelliptic iff

$$g(0,y), g(\pi,y) \notin \pm \mathrm{spec}((-1)^k \partial_z^{2k} + z^{2k})$$

Consider $g(0,y) : S^1 \to \mathbb{C}$.

Example

If $\sin(x) = 0$, then there exists two infinite dimensional representations on $L^2\mathbb{R}$ with

$$\pi_\pm(D) = (-1)^k \partial_z^{2k} + z^{2k} \mp g(x,y) Id_{L^2\mathbb{R}}$$

D is $*$-maximally hypoelliptic iff

$$g(0,y), g(\pi,y) \notin \pm\mathrm{spec}((-1)^k \partial_z^{2k} + z^{2k})$$

Consider $g(0,y) : S^1 \to \mathbb{C}$. Let $w(g(0,y),\lambda)$ be winding number.

Example

If $\sin(x) = 0$, then there exists two infinite dimensional representations on $L^2\mathbb{R}$ with

$$\pi_{\pm}(D) = (-1)^k \partial_z^{2k} + z^{2k} \mp g(x,y) Id_{L^2\mathbb{R}}$$

D is $*$-maximally hypoelliptic iff

$$g(0,y), g(\pi,y) \notin \pm \mathrm{spec}((-1)^k \partial_z^{2k} + z^{2k})$$

Consider $g(0,y) : S^1 \to \mathbb{C}$. Let $w(g(0,y),\lambda)$ be winding number. Then

$$\mathrm{Ind}_a(D) = \sum_\lambda w(g(0,y),\lambda) - w(g(0,y),-\lambda)$$

Example

If $\sin(x) = 0$, then there exists two infinite dimensional representations on $L^2\mathbb{R}$ with

$$\pi_{\pm}(D) = (-1)^k \partial_z^{2k} + z^{2k} \mp g(x,y) Id_{L^2\mathbb{R}}$$

D is $*$-maximally hypoelliptic iff

$$g(0,y), g(\pi,y) \notin \pm \mathrm{spec}((-1)^k \partial_z^{2k} + z^{2k})$$

Consider $g(0,y) : S^1 \to \mathbb{C}$. Let $w(g(0,y),\lambda)$ be winding number. Then

$$\begin{aligned} \mathrm{Ind}_a(D) = & \sum_\lambda w(g(0,y),\lambda) - w(g(0,y),-\lambda) \\ & -w(g(\pi,y),\lambda) + w(g(\pi,y),-\lambda) \end{aligned}$$

Thank you for your attention