A Ruelle dynamical ζ-function for equivariant flows

Peter Hochs

Radboud University

Conference on Noncommutative Geometry and Representation Theory
Athens
10 March 2023

Joint work

Joint work with Hemanth Saratchandran (University of Adelaide):

"A Ruelle dynamical zeta function for equivariant flows", ArXiv:2303.00312.

I The Ruelle dynamical ζ-function

Counting periodic flow curves

In this talk,

- M is a smooth manifold
- φ is a flow on M, i.e. a smooth action by $\mathbb{R}$ on M, **without fixed points**
- $F \to M$ is a vector bundle, and ∇^F a **flat** connection on F.

Counting periodic flow curves

In this talk,

- M is a smooth manifold
- φ is a flow on M, i.e. a smooth action by $\mathbb{R}$ on M, **without fixed points**
- $F \to M$ is a vector bundle, and ∇^F a **flat** connection on F.

The Ruelle dynamical ζ-function is a way to **count periodic flow curves** topologically, "twisted by ∇^F".

Nondegenerate flows

- Consider the **length spectrum**

$$L(\varphi) := \{l > 0;\ \text{there is an } m \in M \text{ such that } \varphi_l(m) = m\}.$$

- For $l \in L(\varphi)$, let $\Gamma_l(\varphi)$ be the set of **closed flow curves of period** l, modulo constant time shifts.

Nondegenerate flows

- Consider the **length spectrum**

$$L(\varphi) := \{l > 0;\ \text{there is an } m \in M \text{ such that } \varphi_l(m) = m\}.$$

- For $l \in L(\varphi)$, let $\Gamma_l(\varphi)$ be the set of **closed flow curves of period** l, modulo constant time shifts.

Definition

The flow φ is **nondegenerate** if for all $l \in L(\varphi)$ and $\gamma \in \Gamma_l(\varphi)$,

$$\ker(1 - T_{\gamma(0)}\varphi_l) = \mathbb{R}\gamma'(0).$$

Lemma

If φ is nondegenerate, then for all $l \in L(\varphi)$, the set $\Gamma_l(\varphi)$ is countable.

The Ruelle dynamical ζ-function

Definition

The **Ruelle dynamical ζ-function** for a nondegenerate φ and ∇^F is

$$R_{\varphi,\nabla^F}(z) := \exp\left(\frac{1}{2}\sum_{l\in L(\varphi)}\frac{e^{-lz}}{l}\sum_{\gamma\in\Gamma_l(\varphi)}\operatorname{sgn}\left(\det\left((1-T_{\gamma(0)}\varphi_l)|_{\gamma'(0)^\perp}\right)\right)T_\gamma^\#\operatorname{tr}(\rho_l(\gamma)^{-1})\right)$$

for $z\in\mathbb{C}$ for which this converges.

Here

- $T_\gamma^\# := \min\{t>0; \gamma(t)=\gamma(0)\}\leq l$ is the **primitive period** of γ
- $\rho_l(\gamma)\colon F_{\gamma(0)}\to F_{\gamma(0)}$ is parallel transport along γ with respect to ∇^F.

The Ruelle dynamical ζ-function

Definition

The **Ruelle dynamical ζ-function** for a nondegenerate φ and ∇^F is

$$R_{\varphi,\nabla^F}(z) :=$$
$$\exp\left(\frac{1}{2}\sum_{l\in L(\varphi)}\frac{e^{-lz}}{l}\sum_{\gamma\in\Gamma_l(\varphi)}\operatorname{sgn}\left(\det\left((1-T_{\gamma(0)}\varphi_l)|_{\gamma'(0)^\perp}\right)\right)T_\gamma^\#\operatorname{tr}(\rho_l(\gamma)^{-1})\right)$$

for $z\in\mathbb{C}$ for which this converges.

Here

- $T_\gamma^\# := \min\{t>0; \gamma(t)=\gamma(0)\}\leq l$ is the **primitive period** of γ
- $\rho_l(\gamma)\colon F_{\gamma(0)}\to F_{\gamma(0)}$ is parallel transport along γ with respect to ∇^F.

Note:

- The terms resemble terms in fixed-point formulas.
- There is a more general definition.

Anosov flows

Let u be the generating vector field of φ.

Definition

The flow φ is **Anosov** if $TM = \mathbb{R}u \oplus E^+ \oplus E^-$, for φ-invariant sub-bundles $E^\pm \subset TM$ such that there is a Riemannian metric on M and $C, c > 0$ such that for all $m \in M$, $v^\pm \in E^\pm_m$ and $t > 0$,

$$\|T_m\varphi_{\pm t}v^\pm\| \leq Ce^{-ct}.$$

If φ is Anosov, then it is nondegenerate.

Anosov flows

Let u be the generating vector field of φ.

Definition

The flow φ is **Anosov** if $TM = \mathbb{R}u \oplus E^+ \oplus E^-$, for φ-invariant sub-bundles $E^\pm \subset TM$ such that there is a Riemannian metric on M and $C, c > 0$ such that for all $m \in M$, $v^\pm \in E^\pm_m$ and $t > 0$,

$$\| T_m \varphi_{\pm t} v^\pm \| \leq C e^{-ct}.$$

If φ is Anosov, then it is nondegenerate.

Proposition (Margulis, 2004)

If φ is Anosov, then $L(\varphi)$ is countable, and there are $C, c > 0$ such that for all $r > 0$,

$$\# \bigcup_{l \leq r} \Gamma_l(\varphi) \leq C e^{cr}.$$

So $R_{\varphi, \nabla^F}(z)$ converges if $\mathrm{Re}(z)$ is large enough.

Properties of the Ruelle dynamical ζ-function for Anosov flows

Theorem (Giulietti–Liverani–Pollicott, 2013; Dyatlov–Zworski, 2016)

If M is compact and orientable and φ is Anosov, then R_{φ,∇^F} has a meromorphic extension to $\mathbb{C}$.

Properties of the Ruelle dynamical ζ-function for Anosov flows

Theorem (Giulietti–Liverani–Pollicott, 2013; Dyatlov–Zworski, 2016)

If M is compact and orientable and φ is Anosov, then R_{φ,∇^F} has a meromorphic extension to $\mathbb{C}$.

Theorem (Dang–Guillarmou–Rivière–Shen, 2020)

Suppose that M is compact and orientable. For Anosov flows, $R_{\varphi,\nabla^F}(0)$, if defined, is invariant under a suitable notion of homotopy.

Properties of the Ruelle dynamical ζ-function for Anosov flows

Theorem (Giulietti–Liverani–Pollicott, 2013; Dyatlov–Zworski, 2016)

If M is compact and orientable and φ is Anosov, then R_{φ,∇^F} has a meromorphic extension to $\mathbb{C}$.

Theorem (Dang–Guillarmou–Rivière–Shen, 2020)

Suppose that M is compact and orientable. For Anosov flows, $R_{\varphi,\nabla^F}(0)$, if defined, is invariant under a suitable notion of homotopy.

Proofs of both results are based on an expression for R_{φ,∇^F} in terms of a distributional **flat trace**; more on this in part **III**.

Geodesic flow

Let X be a Riemannian manifold, and suppose that

$$M = S(TX) := \{v \in TX; \|v\| = 1\}.$$

Let φ be the **geodesic flow** on M:

$$\varphi_t(v) = \left.\frac{d}{ds}\right|_{s=t} \exp_x(sv) \quad \in S(T_{\exp_x(tv)}X),$$

for all $t \in \mathbb{R}$, $x \in X$ and $v \in S(T_xX)$.

Geodesic flow

Let X be a Riemannian manifold, and suppose that

$$M = S(TX) := \{v \in TX; \|v\| = 1\}.$$

Let φ be the **geodesic flow** on M:

$$\varphi_t(v) = \left.\frac{d}{ds}\right|_{s=t} \exp_x(sv) \quad \in S(T_{\exp_x(tv)}X),$$

for all $t \in \mathbb{R}$, $x \in X$ and $v \in S(T_xX)$.

Theorem

If X has negative sectional curvature, then φ is Anosov.

Example: the circle

Let $M = S^1 = \mathbb{R}/\mathbb{Z}$. Define φ by

$$\varphi_t(x + \mathbb{Z}) = x + t + \mathbb{Z},$$

for $t, x \in \mathbb{R}$. Let γ be the unique flow curve up to equivalence

$$\gamma(t) = t + \mathbb{Z}.$$

Then φ is nondegenerate because $\gamma'(0)^\perp = \{0\}$.

Example: the circle

Let $M = S^1 = \mathbb{R}/\mathbb{Z}$. Define φ by

$$\varphi_t(x + \mathbb{Z}) = x + t + \mathbb{Z},$$

for $t, x \in \mathbb{R}$. Let γ be the unique flow curve up to equivalence

$$\gamma(t) = t + \mathbb{Z}.$$

Then φ is nondegenerate because $\gamma'(0)^\perp = \{0\}$.

Let $\nabla^F = d + i\alpha dx$ on $F = M \times \mathbb{C}$, for $\alpha \in \mathbb{R}$. Then

- $L(\varphi) = \mathbb{N}$
- for all $l \in \mathbb{N}$, $\Gamma_l(\varphi) = \{\gamma\}$
- $$\mathrm{sgn}\left(\det\left((1 - T_{\gamma(0)}\varphi_l)|_{\gamma'(0)^\perp}\right)\right) = 1$$
- $T_\gamma^\# = 1$
- $\rho_l(\gamma) = e^{-i\alpha l}$.

Example: the circle

Let $M = S^1 = \mathbb{R}/\mathbb{Z}$. Define φ by

$$\varphi_t(x + \mathbb{Z}) = x + t + \mathbb{Z},$$

for $t, x \in \mathbb{R}$. Let γ be the unique flow curve up to equivalence

$$\gamma(t) = t + \mathbb{Z}.$$

Then φ is nondegenerate because $\gamma'(0)^\perp = \{0\}$.

Let $\nabla^F = d + i\alpha dx$ on $F = M \times \mathbb{C}$, for $\alpha \in \mathbb{R}$. Then

- $L(\varphi) = \mathbb{N}$
- for all $l \in \mathbb{N}$, $\Gamma_l(\varphi) = \{\gamma\}$
- $$\mathrm{sgn}\left(\det\left((1 - T_{\gamma(0)}\varphi_l)|_{\gamma'(0)^\perp}\right)\right) = 1$$
- $T_\gamma^\# = 1$
- $\rho_l(\gamma) = e^{-i\alpha l}$.

So

$$R_{\varphi,\nabla^F}(z) = \exp\left(\frac{1}{2}\sum_{l=1}^{\infty} \frac{e^{-lz-i\alpha l}}{l}\right) = (1 - e^{-z+i\alpha})^{-1/2}.$$

II Equivariant flows

Equivariant flows

From now on, we assume that a unimodular, locally compact group G acts properly on M, such that

- for all $t \in \mathbb{R}$, the map φ_t is equivariant
- $F \to M$ is G-equivariant and ∇^F is G-invariant
- M/G is compact.

Equivariant flows

From now on, we assume that a unimodular, locally compact group G acts properly on M, such that

- for all $t \in \mathbb{R}$, the map φ_t is equivariant
- $F \to M$ is G-equivariant and ∇^F is G-invariant
- M/G is compact.

Example

If X is a Riemannian manifold, on which G acts properly and isometrically, then the lifted action to $M = S(TX)$ has these properties for the geodesic flow.

Example

If X is a compact manifold, and φ_X is a flow on X, then the action by $\pi_1(X)$ on the universal cover M of X has these properties for the lift of φ_X to M.

The g-length spectrum

From now on, fix $g \in G$.

Definition

The g-**length spectrum** of φ is

$$L_g(\varphi) := \{l \neq 0; \text{ there is an } m \in M \text{ such that } \varphi_l(m) = gm\}.$$

The g-length spectrum

From now on, fix $g \in G$.

Definition

The g-**length spectrum** of φ is

$$L_g(\varphi) := \{l \neq 0;\ \text{there is an } m \in M \text{ such that } \varphi_l(m) = gm\}.$$

Now we also include **negative** l.

Example

Consider the manifold $M = \mathbb{R}$, acted on by $G = \mathbb{R}$ by addition. Consider the flow

$$\varphi_t(x) = x + t.$$

Then for all nonzero $g \in \mathbb{R}$,

$$L_g(\varphi) = \{g\}.$$

If $g < 0$, then the g-length spectrum is nonempty because we allow $l < 0$.

g-nondegenerate flows

Definition

We write $\Gamma^g_l(\varphi)$ for the set flow curves γ such that $\gamma(l) = g\gamma(0)$, modulo constant time shifts.

Definition

The flow φ is g-**nondegenerate** if for all $l \in L_g(\varphi)$ and $\gamma \in \Gamma^g_l(\varphi)$,

$$\ker(1 - T_{\gamma(0)}\varphi_l \circ g^{-1}) = \mathbb{R}\gamma'(0).$$

g-nondegenerate flows

Definition

We write $\Gamma_l^g(\varphi)$ for the set flow curves γ such that $\gamma(l) = g\gamma(0)$, modulo constant time shifts.

Definition

The flow φ is g-**nondegenerate** if for all $l \in L_g(\varphi)$ and $\gamma \in \Gamma_l^g(\varphi)$,

$$\ker(1 - T_{\gamma(0)}\varphi_l \circ g^{-1}) = \mathbb{R}\gamma'(0).$$

Lemma

If φ is g-nondegenerate, then for all $l \in L_g(\varphi)$, the set $\Gamma_l^g(\varphi)$ is countable.

An equivariant primitive period?

Question: what is the most useful equivariant generalisation of the primitive period

$$T_\gamma^\# = \min\{t > 0; \gamma(t) = \gamma(0)\}?$$

An equivariant primitive period?

Question: what is the most useful equivariant generalisation of the primitive period

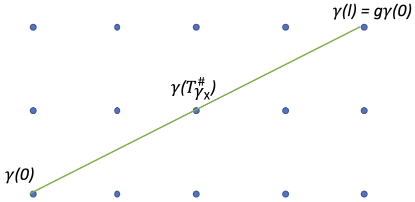
$$T_\gamma^{\#} = \min\{t > 0; \gamma(t) = \gamma(0)\}?$$

Example

If $\gamma(l) = g\gamma(0)$, then one could define

$$T_\gamma^{g} := \min\{t > 0; \gamma(t) = g\gamma(0)\}.$$

However, if M is the universal cover of a manifold X, and $G = \pi_1(X)$, this does not encode the primitive period of a flow curve in X.



The g-primitive period

Because M/G is compact and the action by G is proper, there is a $\chi \in C_c^\infty(M)$ such that for all $m \in M$,

$$\int_G \chi(xm)\, dx = 1.$$

Definition

Let $\gamma\colon \mathbb{R} \to M$ be any smooth curve. Let $I_\gamma \subset \mathbb{R}$ be an interval such that $\gamma|_{I_\gamma}$ is a bijection onto its image, up to sets of measure zero. Then the χ-**primitive period** of γ is

$$T_\gamma^\chi := \int_{I_\gamma} \chi(\gamma(t))\, dt.$$

Note that T_γ^χ also depends on I_γ.

The g-primitive period in the compact case

Suppose that G is **compact** (e.g. trivial), and normalise dx so that $\mathrm{vol}(G) = 1$. Then M is also compact. And $\chi \equiv 1$ satisfies

$$\int_G \chi(xm)\, dx = 1$$

for all $m \in M$.

The g-primitive period in the compact case

Suppose that G is **compact** (e.g. trivial), and normalise dx so that $\mathrm{vol}(G) = 1$. Then M is also compact. And $\chi \equiv 1$ satisfies

$$\int_G \chi(xm)\, dx = 1$$

for all $m \in M$.

If γ is a periodic curve in M, then we can take $I_\gamma = [0, T_\gamma^\#]$. So

$$T_\gamma^\chi = \int_{I_\gamma} \chi(\gamma(t))\, dt = \int_0^{T_\gamma^\#} 1\, dt = T_\gamma^\#.$$

The g-primitive period for geodesic flow on universal covers

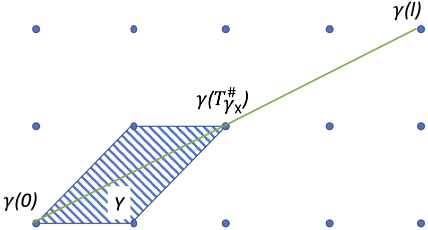
Let X be a compact Riemannian manifold, $\tilde{X}$ its universal cover, and $M = S(T\tilde{X})$, acted on by $G = \pi_1(X)$.

The g-primitive period for geodesic flow on universal covers

Let X be a compact Riemannian manifold, $\tilde{X}$ its universal cover, and $M = S(T\tilde{X})$, acted on by $G = \pi_1(X)$.

If X has negative sectional curvature, then the conjugacy class of an element of G contains a unique closed geodesic γ_X. Let l be the period of such a γ_X. Then $\gamma([0, T^{\#}_{\gamma_X}])$ lies in a fundamental domain $Y \subset M$ for the action by G. By approximating 1_Y be smooth functions χ, we can make T^{χ}_{γ} arbitrarily close to

$$\int_{\mathbb{R}} 1_Y(\gamma(t))\, dt = T^{\#}_{\gamma_X}.$$



The equivariant Ruelle ζ-function

Suppose that

- $L_g(\varphi)$ is countable
- φ is g-nondegenerate
- if $Z < G$ is the centraliser of g, then G/Z has a G-invariant measure $d(hZ)$.

Definition

The **equivariant Ruelle dynamical ζ-function** for g, φ and ∇^F is

$$2 \log R^g_{\varphi,\nabla^F}(z) := \int_{G/Z} \sum_{l \in L_g(\varphi)} \frac{e^{-lz}}{l} \sum_{\gamma \in \Gamma^g_l(\varphi)} \operatorname{sgn}\left(\det\left((1 - T_{\gamma(0)}\varphi_l \circ g^{-1})|_{\gamma'(0)^\perp}\right)\right) T^{\chi}_{h\gamma} \operatorname{tr}(g \circ \rho_l(\gamma)^{-1})\, d(hZ),$$

for $z \in \mathbb{C}$ for which this converges.

Well-definedness

$$2 \log R^g_{\varphi,\nabla^F}(z) =$$
$$\int_{G/Z} \sum_{l \in L_g(\varphi)} \frac{e^{-lz}}{l} \sum_{\gamma \in \Gamma^g_l(\varphi)} \operatorname{sgn}\left(\det\left((1 - T_{\gamma(0)}\varphi_l \circ g^{-1})|_{\gamma'(0)^\perp}\right)\right)$$
$$T^\chi_{h\gamma} \operatorname{tr}(g \circ \rho_l(\gamma)^{-1})\, d(hZ).$$

Lemma

The integrand is right Z-invariant, so indeed defines a function on G/Z.

Well-definedness

$$2 \log R^g_{\varphi,\nabla^F}(z) =$$
$$\int_{G/Z} \sum_{l \in L_g(\varphi)} \frac{e^{-lz}}{l} \sum_{\gamma \in \Gamma^g_l(\varphi)} \operatorname{sgn}\left(\det\left((1 - T_{\gamma(0)}\varphi_l \circ g^{-1})|_{\gamma'(0)^\perp}\right)\right)$$
$$T^\chi_{h\gamma} \operatorname{tr}(g \circ \rho_l(\gamma)^{-1})\, d(hZ).$$

Lemma

The integrand is right Z-invariant, so indeed defines a function on G/Z.

Proposition

The function R^g_{φ,∇^F} is independent of the cutoff function χ and the interval $I_{h\gamma}$ in the definition of $T^\chi_{h\gamma}$.

The classical Ruelle ζ-function

Lemma

If G is trivial and M is odd-dimensional, then

$$R^{e}_{\varphi,\nabla^F} = |R_{\varphi,\nabla^F}|.$$

Flows on universal covers

Proposition

If M is the universal cover of a compact manifold X, acted on by $G = \pi_1(X)$, and φ, F and ∇^F are pullbacks of corresponding data φ_X, F_X and ∇^{F_X} on X, then

$$R_{\varphi_X, \nabla^{F_X}} = \prod_{(g)} R^g_{\varphi, \nabla^F},$$

where the product is over the conjugacy classes in $\pi_1(X)$.

Flows on universal covers

Proposition

If M is the universal cover of a compact manifold X, acted on by $G = \pi_1(X)$, and φ, F and ∇^F are pullbacks of corresponding data φ_X, F_X and ∇^{F_X} on X, then

$$R_{\varphi_X, \nabla^{F_X}} = \prod_{(g)} R^g_{\varphi, \nabla^F},$$

where the product is over the conjugacy classes in $\pi_1(X)$.

Lemma

In the setting of the previous proposition, if $X = S(TZ)$ for Z with negative sectional curvature, φ_X is geodesic flow, and F_X and ∇^{F_X} are associated to a representation ρ of $\pi_1(Z)$, then

$$R^g_{\varphi, \nabla^F}(z) = \exp\left(\frac{1}{2}\frac{e^{-lz}}{l} T^{\#}_{\gamma} \operatorname{tr}(\rho(g))\right),$$

where γ is the closed geodesic in (g) and l is its period.

Example: the circle

Let $M = \mathbb{R}/\mathbb{Z}$. Define

$$\varphi_t(x + \mathbb{Z}) = x + t + \mathbb{Z},$$

and

$$\gamma(t) = t + \mathbb{Z}.$$

Example: the circle

Let $M = \mathbb{R}/\mathbb{Z}$. Define

$$\varphi_t(x + \mathbb{Z}) = x + t + \mathbb{Z},$$

and

$$\gamma(t) = t + \mathbb{Z}.$$

Let $G = \mathbb{R}/\mathbb{Z}$, acting on M in the natural way. Let $g = r + \mathbb{Z} \in G$.

Example: the circle

Let $M = \mathbb{R}/\mathbb{Z}$. Define

$$\varphi_t(x+\mathbb{Z}) = x + t + \mathbb{Z},$$

and

$$\gamma(t) = t + \mathbb{Z}.$$

Let $G = \mathbb{R}/\mathbb{Z}$, acting on M in the natural way. Let $g = r + \mathbb{Z} \in G$. If $r \notin \mathbb{Z}$, then

- $L_g(\varphi) = r + \mathbb{Z}$
- for all $l \in L_g(\varphi)$, $\Gamma_l^g(\varphi) = \{\gamma\}$
- $$\operatorname{sgn}\left(\det\left((1 - T_{\gamma(0)}\varphi_l \circ g^{-1})|_{\gamma'(0)^\perp}\right)\right) = 1$$
- $T_{h\gamma}^{\chi} = 1$, for $\chi \equiv 1$
- $g \circ \rho_l(\gamma)^{-1} = e^{i\alpha l}$, for $F = M \times \mathbb{C}$ and $\nabla^F = d + i\alpha dx$.

So

$$R^g_{\varphi,\nabla^F}(z) = \exp\left(\frac{1}{2}\sum_{n\in\mathbb{Z}} \frac{e^{-|n+r|z + i\alpha(n+r)}}{|n+r|}\right).$$

Example: the line

Let $M = \mathbb{R}$. Define

$$\varphi_t(x) = x + t,$$

and

$$\gamma(t) = t.$$

Example: the line

Let $M = \mathbb{R}$. Define

$$\varphi_t(x) = x + t,$$

and

$$\gamma(t) = t.$$

Let $G = \mathbb{R}$, acting on M by addition. Let $g \in G$.

Example: the line

Let $M = \mathbb{R}$. Define

$$\varphi_t(x) = x + t,$$

and

$$\gamma(t) = t.$$

Let $G = \mathbb{R}$, acting on M by addition. Let $g \in G$. Then

- $L_g(\varphi) = \{g\}$
- $\Gamma_g^g(\varphi) = \{\gamma\}$
- $$\mathrm{sgn}\left(\det\left((1 - T_{\gamma(0)}\varphi_g \circ g^{-1})|_{\gamma'(0)^\perp}\right)\right) = 1$$
- $$T^\chi_{h\gamma} = \int_{\mathbb{R}} \chi(t+h)\, dt = 1$$
- $g \circ \rho_g(\gamma)^{-1} = e^{i\alpha g}$, for $F = M \times \mathbb{C}$ and $\nabla^F = d + i\alpha dx$.

So, without convergence issues,

$$R^g_{\varphi,\nabla^F}(z) = \exp\left(\frac{1}{2}\frac{e^{-|g|z + i\alpha g}}{|g|}\right).$$

III A trace formula

A trace formula for the Ruelle ζ-function

Let Φ be the lift of φ to $\bigwedge^* T^*M \otimes F$ given by

$$\Phi_t := \wedge T\varphi^*_{-t} \otimes \tau_t^{\nabla^F},$$

where $\tau_t^{\nabla^F}$ is parallel transport in F with respect to ∇^F along flow curves.
Let N be the number operator on differential forms on M.

A trace formula for the Ruelle ζ-function

Let Φ be the lift of φ to $\bigwedge^* T^*M \otimes F$ given by

$$\Phi_t := \wedge T\varphi^*_{-t} \otimes \tau_t^{\nabla^F},$$

where $\tau_t^{\nabla^F}$ is parallel transport in F with respect to ∇^F along flow curves. Let N be the number operator on differential forms on M.

Corollary (Of Guillemin's trace formula, 1977)

If M is compact, φ is Anosov and $\mathrm{Re}(z)$ is large,

$$R_{\varphi,\nabla^F}(z) = \exp\left(-\frac{1}{2}\int_0^\infty \mathrm{Tr}^\flat\left((-1)^N N\Phi_t^*\right)\frac{e^{-tz}}{t}\,dt\right).$$

Here $\mathrm{Tr}^\flat$ is defined by "integrating" distributional Schwartz kernels over the diagonal. (The wave front set of Φ_t^* is disjoint from the conormal bundle because of Anosov/nondegeneracy condition.)

A trace formula for the Ruelle ζ-function

Let Φ be the lift of φ to $\bigwedge^* T^*M \otimes F$ given by

$$\Phi_t := \wedge T\varphi^*_{-t} \otimes \tau_t^{\nabla^F},$$

where $\tau_t^{\nabla^F}$ is parallel transport in F with respect to ∇^F along flow curves. Let N be the number operator on differential forms on M.

Corollary (Of Guillemin's trace formula, 1977)

If M is compact, φ is Anosov and $\mathrm{Re}(z)$ is large,

$$R_{\varphi,\nabla^F}(z) = \exp\left(-\frac{1}{2}\int_0^\infty \mathrm{Tr}^\flat\left((-1)^N N\Phi_t^*\right)\frac{e^{-tz}}{t}\,dt\right).$$

Here $\mathrm{Tr}^\flat$ is defined by "integrating" distributional Schwartz kernels over the diagonal. (The wave front set of Φ_t^* is disjoint from the conormal bundle because of Anosov/nondegeneracy condition.)

Used in proofs of meromorphic extension and homotopy invariance for Anosov flows.

The flat g-trace

Definition

If T is a G-equivariant operator on smooth sections of a G-vector bundle over M, with Schwartz kernel K, then the **flat g-trace** of T is

$$\mathrm{Tr}_g^\flat(T) := \int_{G/Z} \int_M \chi(hgh^{-1}m)\, \mathrm{tr}\big(hgh^{-1}K(hg^{-1}h^{-1}m, m)\big)\, dm\, d(hZ),$$

when defined and convergent.

Special cases:

- If G and M are compact, then
$$\mathrm{Tr}_g^\flat(T) = \mathrm{Tr}^\flat(g \circ T).$$
- If K is smooth, then we recover the orbital integral trace
$$\mathrm{Tr}_g^\flat(T) = \mathrm{Tr}_g(T),$$
used in various places.

An expression for the equivariant Ruelle ζ-function

Theorem (H-Saratchandran, 2023)

If φ is g-nondegenerate and $\mathrm{Re}(z)$ *is large,*

$$R^{g}_{\varphi,\nabla^F}(z) = \exp\left(-\frac{1}{2}\int_{\mathbb{R}\setminus\{0\}} \mathrm{Tr}^{\flat}_{g}\left((-1)^N N\Phi_t^*\right)\frac{e^{-|t|z}}{|t|}\,dt\right).$$

This follows from an equivariant generalisation of Guillemin's trace formula.

An expression for the equivariant Ruelle ζ-function

Theorem (H-Saratchandran, 2023)

If φ is g-nondegenerate and $\mathrm{Re}(z)$ *is large,*

$$R^g_{\varphi,\nabla^F}(z) = \exp\left(-\frac{1}{2}\int_{\mathbb{R}\setminus\{0\}} \mathrm{Tr}^\flat_g\left((-1)^N N\Phi_t^*\right)\frac{e^{-|t|z}}{|t|}\,dt\right).$$

This follows from an equivariant generalisation of Guillemin's trace formula.

Corollaries:

- independence of R^g_{φ,∇^F} of χ and I_γ
- decomposition of the classical Ruelle ζ-function in terms of conjugacy classes in the fundamental group.

IV An equivariant Fried conjecture

The Fried conjecture

Suppose that M is compact, oriented, and Riemanian. Let $\Delta_F := (\nabla^F)^*\nabla^F + \nabla^F(\nabla^F)^*$.

Definition (Ray–Singer, 1971)

The **analytic torsion** of M, twisted by ∇^F, is

$$T_{\nabla^F}(M) := \exp\left(-\frac{1}{2}\,\frac{d}{ds}\bigg|_{s=0} \mathrm{Tr}\big((-1)^N N(\Delta_F|_{\ker(\Delta_F)^\perp})^{-s}\big)\right).$$

The Fried conjecture

Suppose that M is compact, oriented, and Riemanian. Let $\Delta_F := (\nabla^F)^* \nabla^F + \nabla^F (\nabla^F)^*$.

Definition (Ray–Singer, 1971)

The **analytic torsion** of M, twisted by ∇^F, is

$$T_{\nabla^F}(M) := \exp\left(-\frac{1}{2} \frac{d}{ds}\bigg|_{s=0} \mathrm{Tr}\left((-1)^N N (\Delta_F|_{\ker(\Delta_F)^\perp})^{-s} \right) \right).$$

Conjecture (Fried, 1987)

If $\ker(\Delta_F) = 0$, *then for a large class of flows,* $R_{\varphi, \nabla^F}(z)$ *extends to* $z = 0$ *and*

$$T_{\nabla^F}(M) = |R_{\varphi, \nabla^F}(0)|.$$

The Fried conjecture

Suppose that M is compact, oriented, and Riemanian. Let $\Delta_F := (\nabla^F)^*\nabla^F + \nabla^F(\nabla^F)^*$.

Definition (Ray–Singer, 1971)

The **analytic torsion** of M, twisted by ∇^F, is

$$T_{\nabla^F}(M) := \exp\left(-\frac{1}{2}\frac{d}{ds}\Big|_{s=0} \mathrm{Tr}\big((-1)^N N(\Delta_F|_{\ker(\Delta_F)^\perp})^{-s}\big)\right).$$

Conjecture (Fried, 1987)

If $\ker(\Delta_F) = 0$, *then for a large class of flows,* $R_{\varphi,\nabla^F}(z)$ *extends to* $z = 0$ *and*

$$T_{\nabla^F}(M) = |R_{\varphi,\nabla^F}(0)|.$$

Proved in various cases by Bismut, Dang–Guillarmou–Rivière–Shen, Fried, Moscovici–Stanton, Müller, Sànchez-Morgado, Shen, Shen–Yu, Spilioti, Wotzke, Yamaguchi, . . .

An equivariant Fried conjecture

We can define **equivariant analytic torsion** $T^g_{\nabla^F}(M)$ if M/G is compact, by replacing the operator trace by Tr_g. (Studied by Bismut, Bismut–Goette, Deitmar, Köhler, Lott, Lott–Rothenberg, Lück, for compact G; Lott, Mathai, H-Saratchandran, for noncompact G.)

An equivariant Fried conjecture

We can define **equivariant analytic torsion** $T^g_{\nabla^F}(M)$ if M/G is compact, by replacing the operator trace by Tr_g. (Studied by Bismut, Bismut–Goette, Deitmar, Köhler, Lott, Lott–Rothenberg, Lück, for compact G; Lott, Mathai, H-Saratchandran, for noncompact G.)

Question (Equivariant Fried problem/conjecture)

If $\dim(M)$ *is odd and* $\ker_{L^2}(\Delta_F) = 0$, *under what further conditions is*

$$T^g_{\nabla^F}(M) = R^g_{\varphi, \nabla^F}(0)?$$

An equivariant Fried conjecture

We can define **equivariant analytic torsion** $T^g_{\nabla^F}(M)$ if M/G is compact, by replacing the operator trace by Tr_g. (Studied by Bismut, Bismut–Goette, Deitmar, Köhler, Lott, Lott–Rothenberg, Lück, for compact G; Lott, Mathai, H-Saratchandran, for noncompact G.)

Question (Equivariant Fried problem/conjecture)

If $\dim(M)$ *is odd and* $\ker_{L^2}(\Delta_F) = 0$, *under what further conditions is*

$$T^g_{\nabla^F}(M) = R^g_{\varphi,\nabla^F}(0)?$$

- We don't need an absolute value now, because we allow $l < 0$.
- If $\dim(M)$ is even, then $T^g_{\nabla^F}(M) = 1$, whereas $R^g_{\varphi,\nabla^F}(0)$ may be different from 1.

A non-example

For the lift to the universal cover of geodesic flow on the sphere bundle of a compact Riemannian manifold, then the equivariant Fried conjecture does **not** hold.

A non-example

For the lift to the universal cover of geodesic flow on the sphere bundle of a compact Riemannian manifold, then the equivariant Fried conjecture does **not** hold.

Already in Fried's original result for hyperbolic manifolds,

$$T^e_{\nabla^F}(M) \neq 1 = R^e_{\varphi,\nabla^F}(0).$$

But Fried proved that (up to regularisation)

$$T_{\nabla^F}(M) = \prod_{(g)} T^g_{\nabla^F}(M) = \prod_{(g)} R^g_{\varphi,\nabla^F}(0) = |R_{\varphi,\nabla^F}(0)|.$$

The circle

For the circle acting on itself, we saw

$$R^g_{\varphi,\nabla^F}(z) = \exp\left(\frac{1}{2}\sum_{n\in\mathbb{Z}} \frac{e^{-|n+r|z+i\alpha(n+r)}}{|n+r|}\right).$$

Now

$$T^g_{\nabla^F}(M) = \exp\left(\frac{1}{2}\sum_{n\in\mathbb{Z}} \frac{e^{i\alpha(n+r)}}{|n+r|}\right) = R^g_{\varphi,\nabla^F}(0).$$

The line

For the real line acting on itself, we saw

$$R^g_{\varphi,\nabla^F}(z) = \exp\left(\frac{1}{2}\frac{e^{-|g|z+i\alpha g}}{|g|}\right).$$

Now

$$T^g_{\nabla^F}(M) = \exp\left(\frac{1}{2}\frac{e^{i\alpha g}}{|g|}\right) = R^g_{\varphi,\nabla^F}(0).$$

Geodesic flow in $\mathbb{R}^n$

For geodesic flow on $S(T\mathbb{R}^n)$, acted on by a discrete subgroup of the Euclidean motion group, we have for certain g,

$$T^g_{\nabla^F}(M) = R^g_{\varphi,\nabla^F}(0).$$

Geodesic flow in $\mathbb{R}^n$

For geodesic flow on $S(T\mathbb{R}^n)$, acted on by a discrete subgroup of the Euclidean motion group, we have for certain g,

$$T^g_{\nabla^F}(M) = R^g_{\varphi,\nabla^F}(0).$$

Now the classical Ruelle ζ-function is not defined, because there are no closed geodesics.

Geodesic flow on spheres

For geodesic flow on $M = S(TS^n)$, acted on by $G = \mathrm{SO}(n+1)$, and g regular, and the trivial connection d on $M \times \mathbb{C}$,

- $L_g(\varphi)$ is countable
- the flow is g-nondegenerate
- we can compute $R^g_{\varphi,\nabla^F}(z)$,

at least for $n = 2$ and $n = 3$.

Geodesic flow on spheres

For geodesic flow on $M = S(TS^n)$, acted on by $G = \mathrm{SO}(n+1)$, and g regular, and the trivial connection d on $M \times \mathbb{C}$,

- $L_g(\varphi)$ is countable
- the flow is g-nondegenerate
- we can compute $R^g_{\varphi,\nabla^F}(z)$,

at least for $n = 2$ and $n = 3$.

But

- $\ker \Delta_F = H^*(M) \neq 0$
- R^g_{φ,∇^F} does not extend to zero.

So the conditions of the equivariant Fried conjecture do not hold.

Geodesic flow on spheres

For geodesic flow on $M = S(TS^n)$, acted on by $G = \mathrm{SO}(n+1)$, and g regular, and the trivial connection d on $M \times \mathbb{C}$,

- $L_g(\varphi)$ is countable
- the flow is g-nondegenerate
- we can compute $R^g_{\varphi,\nabla^F}(z)$,

at least for $n = 2$ and $n = 3$.

But

- $\ker \Delta_F = H^*(M) \neq 0$
- R^g_{φ,∇^F} does not extend to zero.

So the conditions of the equivariant Fried conjecture do not hold.

Now the classical Ruelle ζ-function is not defined, because there are uncountably many closed geodesics.

A superficial similarity

Modulo suitable regularisation and interpretation,

$$\text{“}R^g_{\varphi,\nabla^F}(0) = \exp\left(-\frac{1}{2}\int_{\mathbb{R}\setminus\{0\}} \mathrm{Tr}^\flat_g\left((-1)^N N\Phi_t^*\right)\frac{1}{|t|}\,dt\right)\text{”},$$

and

$$\text{“}T^g_{\nabla^F}(M) = \exp\left(-\frac{1}{2}\int_0^\infty \mathrm{Tr}_g\left((-1)^N N e^{-t\Delta_F}\right)\frac{1}{t}\,dt\right)\text{”}.$$

Thank you