Hypoelliptic Laplacian and the trace formula

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Non commutative geometry and representation theory

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- If $\square^X = D^{X,2}$, deformation constructed via deformation of D^X.

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- Here, we will concentrate on the trace formula.

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- L_b^X geometric Fokker-Planck operator.

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- $g = 1$: Euler characteristic.

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3. By making $b \to +\infty$, do we obtain Selberg's trace formula ? (extension of Poisson formula).
4. Is Selberg's trace formula an index formula?

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- Left-hand side: spectral, right-hand side: closed geodesics.

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- Right-hand side orbital integrals for $G = \mathrm{SL}_2\left(\mathbf{R}\right)$.

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- Make the heat kernel g lift to $\left(\Omega^{\bullet}(E, \mathbf{R}), d^{E}\right)$.

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Example

$G = \mathrm{SL}_2(\mathbf{R})$, $K = S^1$, X upper half-plane, $TX \oplus N$ of dimension 3.

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- Both constrictions involve Dirac operators.

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- $D^{\mathfrak{g}} = \underline{d}^{\mathfrak{g}} + \underline{d}^{\mathfrak{g}*}$ Dirac operator acts on $C^{\infty}\left(\mathfrak{g}, \Lambda\left(\mathfrak{g}^*_{\mathbf{C}}\right)\right)$.
- $\frac{1}{2}\left[\underline{d}^{\mathfrak{g}}, \underline{d}^{\mathfrak{g}*}\right] = H^{\mathfrak{g}} + N^{\Lambda^{\bullet}(\mathfrak{g}^*)}$.

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Remark

Using the fiberwise Bargmann isomorphism, $\mathcal{L}_b^X$ acts on

$$C^\infty\left(X, S^\bullet\left(T^*X \oplus N^*\right) \otimes \Lambda^\bullet \left(T^*X \oplus N^*\right)_{\mathbf{C}}\right).$$

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$$\mathcal{L}_b^X = \frac{1}{2}\left|\left[Y^N, Y^{TX}\right]\right|^2 + \underbrace{\frac{1}{2b^2}\left(-\Delta^{TX\oplus N} + |Y|^2 - n\right)}_{\text{Harmonic oscillator of } TX\oplus N} + \frac{N^{\Lambda^\bullet(T^*X\oplus N^*)}}{b^2}$$

$$+\frac{1}{b}\left(\underbrace{\nabla_{Y^{TX}}}_{\text{geodesic flow}} + \widehat{c}\left(\mathrm{ad}\left(Y^{TX}\right)\right) - c\left(\mathrm{ad}\left(Y^{TX}\right) + i\theta\mathrm{ad}\left(Y^N\right)\right)\right).$$

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- $b \to +\infty$, geodesic f. $\nabla_{Y^{TX}}$ dominates $\Rightarrow$ closed geodesics.

The case of locally symmetric spaces

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- $Z = \Gamma \setminus X$ compact locally symmetric.

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For $t > 0, b > 0$,

$$\mathrm{Tr}^{C^\infty(Z,E)}\left[\exp\left(-t\left(C^{\mathfrak{g},Z} - c\right)/2\right)\right] = \mathrm{Tr_s}\left[\exp\left(-t\mathcal{L}_b^Z\right)\right].$$

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•$TZ \oplus N^Z$ bigger than the tangent bundle.

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2. The above identity splits as an identity of orbital integrals.

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Remark

The proof uses the fact that $\mathrm{Tr}^{[\gamma]}$ is a trace on the algebra of G-invariants smooth kernels on X with Gaussian decay.

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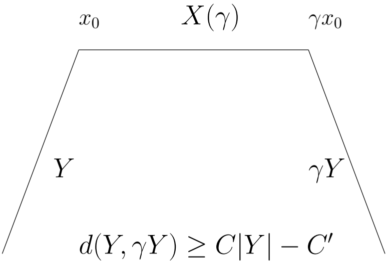

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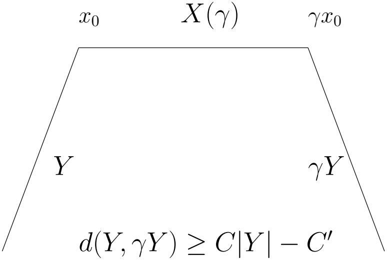

$$p_t^X(x, x') \leq C \exp(-C' d^2(x, x')).$$

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- $\gamma = e^a k^{-1}, a \in \mathfrak{p}, k \in K, \mathrm{Ad}(k) a = a$.
- $Z(\gamma)$ centralizer of γ, $\mathfrak{z}(\gamma) = \mathfrak{p}(\gamma) \oplus \mathfrak{k}(\gamma)$ Lie algebra of $Z(\gamma)$.

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Local index techniques play fundamental role in the evaluation.

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Definition

$$\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right) = \frac{1}{\left|\det\left(1-\operatorname{Ad}(\gamma)\right)|_{\mathfrak{z}_0^\perp}\right|^{1/2}} \frac{\widehat{A}\left(\operatorname{ad}\left(Y_0^{\mathfrak{k}}\right)|_{\mathfrak{p}(\gamma)}\right)}{\widehat{A}\left(\operatorname{ad}\left(Y_0^{\mathfrak{k}}\right)_{\mathfrak{k}(\gamma)}\right)}$$

$$\left[\frac{1}{\det\left(1-\operatorname{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{z}_0^\perp(\gamma)}} \frac{\det\left(1-\operatorname{Ad}\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right)|_{\mathfrak{k}_0^\perp(\gamma)}}{\det\left(1-\operatorname{Ad}\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right)|_{\mathfrak{p}_0^\perp(\gamma)}}\right]^{1/2}.$$

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- By making $b \to +\infty$, $p_t(a) = \frac{1}{\sqrt{2\pi t}}\exp\left(-a^2/2t\right)$.
- $\exp\left(t\Delta^{\mathbf{R}}/2\right)(a) = \frac{1}{\sqrt{2\pi t}}\exp\left(-a^2/2t\right)$ is an index formula!

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- The hypoelliptic Laplacian gives a role to mass in classical math!

Langevin (C.R. de l'Académie des Sciences 1908)

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Une particule comme celle que nous considérons, grande par rapport à la distance moyenne des molécules du liquide, et se mouvant par rapport à celui-ci avec la vitesse ξ subit une résistance visqueuse égale à $-6\pi\mu a\xi$ d'après la formule de Stokes. En réalité, cette valeur n'est qu'une moyenne, et en raison de l'irrégularité des chocs des molécules environnantes, l'action du fluide sur la particule oscille autour de la valeur précédente, de sorte que l'équation du mouvement est, dans la direction x,

$$m\frac{d^2x}{dt^2} = -6\pi\mu a\frac{dx}{dt} + X. \tag{3}$$

Sur la force complémentaire X nous savons qu'elle est indifféremment positive et négative, et sa grandeur est telle qu'elle maintient l'agitation de la particule que, sans elle, la résistance visqueuse finirait par arrêter.

- P. Langevin, *Sur la théorie du mouvement brownien*, C. R. Acad. Sci. Paris **146** (1908), 530–533.
- J.-M. Bismut, *Hypoelliptic Laplacian and orbital integrals*, Annals of Mathematics Studies, vol. 177, Princeton University Press, Princeton, NJ, 2011. MR 2828080

Ευχαριστώ πολύ!