From the prolate spheroid to zeta spectrum

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A. Connes, C. Consani, *Weil positivity and trace formula, the archimedean place*. Selecta Math. (N.S.) 27 (2021) no 4.

A. Connes, C. Consani, *Quasi-inner functions and local factors*, Journal of Number Theory, 226 , pp. 139–167, 2021.

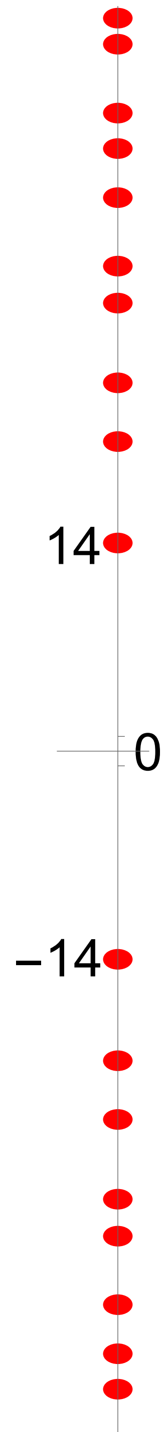
A. Connes, C. Consani, *Spectral triples and ζ-cycles*. (2021).

A. Connes, H. Moscovici, *The UV prolate spectrum matches the zeros of zeta*. Proc. Natl. Acad. Sci. U.S.A. 119, e2123174119 (2022).

- Using trace formula for Weil positivity (joint work with C. Consani)

- minuscule eigenvalues, prolate functions and low lying zeros of zeta (joint work with C. Consani)

- Prolate wave operator and the ultraviolet part of the spectrum (joint work with H. Moscovici).


14
0
−14

Ultraviolet

$$N(E) = \frac{E}{2\pi}\log\frac{E}{2\pi} - \frac{E}{2\pi} + O(\log(E))$$

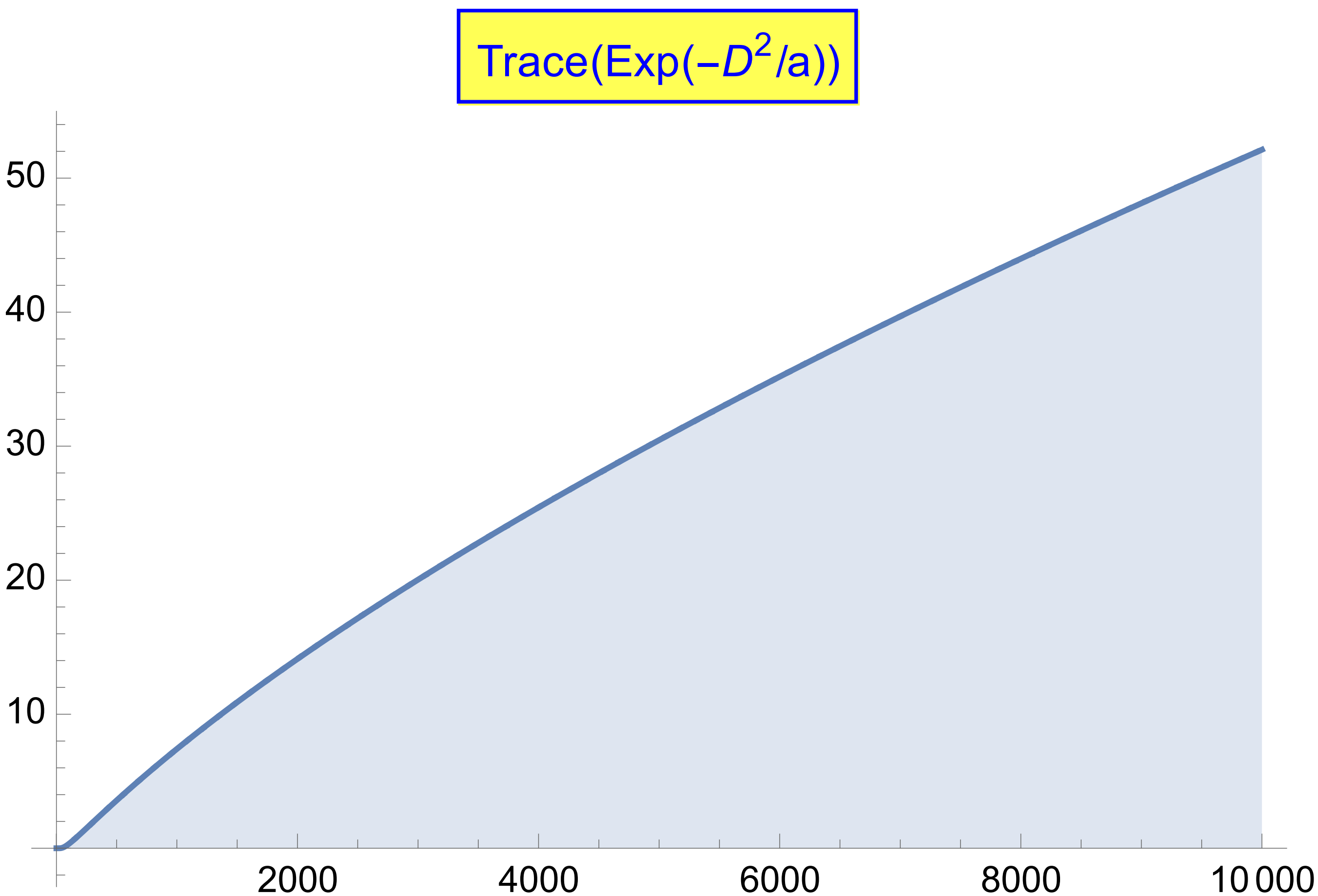
$$\mathrm{Tr}(\exp(-tD^2)) = \frac{\log\left(\frac{1}{t}\right)}{4\sqrt{\pi}\sqrt{t}} - \frac{(\log 4\pi + \frac{1}{2}\gamma)}{2\sqrt{\pi}\sqrt{t}}$$

$$+O\left(\log\left(\frac{1}{t}\right)\right)$$

In fact, more precisely

$$\frac{\log\left(\frac{1}{t}\right)}{4\sqrt{\pi}\sqrt{t}} - \frac{\gamma}{4\sqrt{\pi}\sqrt{t}} - \frac{\log(4\pi)}{2\sqrt{\pi}\sqrt{t}} + \frac{7}{4} + \frac{\sqrt{t}}{24\sqrt{\pi}} + \frac{9\,t}{16} + \dots$$

Infrared



Trace(Exp(−D^2/a))
50
40
30
20
10
2000
4000
6000
8000
10 000

Riemann-Weil explicit formula

Function f on $\mathbb{R}_+^*$, dual group $\mathbb{R}$,

$$\widehat{f}(s) := \int f(u)u^{-is}d^*u, \quad d^*u = du/u$$

$$\widehat{f}(i/2) - \sum_{\frac{1}{2}+is\in Z} \widehat{f}(s) + \widehat{f}(-i/2) =$$

$$= \sum_v W_v(f)$$

$$W_{\mathbb{R}}(f) := (\log 4\pi + \gamma) f(1) +$$

$$+ \int_1^\infty \Big(f(x) + f(x^{-1}) - 2x^{-1/2} f(1) \Big) \frac{x^{1/2}}{x - x^{-1}} d^*x$$

$$W_p(f) = (\log p) \sum_{m=1}^{\infty} p^{-m/2} \Big(f(p^m) + f(p^{-m}) \Big)$$

Delicate distributional principal values

$$RH \iff \sum_v W_v(f * f^*) \leq 0,$$

$$\forall f \in C_c^\infty(\mathbb{R}_+^*),\ \widehat{f}(\pm i/2) = 0$$

Trace formula
Geometry, Hilbert space

S finite set of places containing ∞

$$X_S := \left(\prod_{v\in S} \mathbb{Q}_v\right) / \{\pm \prod p_v^{n_v}\} \to L^2(X_S)$$

$\vartheta(h) =$ scaling action, $\widehat{P} := \mathbb{F}_S \, P \, \mathbb{F}_S^{-1}$

$$\mathrm{Tr}\left(\left(P + \widehat{P} - 1\right) \vartheta(h)\right) = \sum_{v\in S} W_v(h)$$

$P_\lambda =$ projection on the subspace

$$\{\xi \mid \xi(x) = 0 \ \ \forall x, \ \ |x| > \lambda\}$$

Pair of projections, angle, $\alpha = \angle(P_\lambda, \widehat{P}_\lambda)$

Archimedean place, prolate spheroidal wave functions ψ_m

$$P_\lambda \mathbb{F}_{e_\mathbb{R}} P_\lambda \psi_m = (-1)^m \chi(\mu, m) P_\lambda \psi_m$$

$$P_\lambda \widehat{P}_\lambda P_\lambda \psi_m = \chi(\mu, m)^2 P_\lambda \psi_m, \quad \mu = \lambda^2$$

$$\cos^2(\alpha_m) = \chi(\mu, m)^2$$

S_λ = Projection on Sonin's space, Orthogonal to both P_λ and $\widehat{P_\lambda}$. For $\lambda = 1$,

$$\mathrm{tr}(\vartheta(h)\mathbf{S}) = -W_{\mathbb{R}}(h) + \int h(\rho)\epsilon(\rho)d^*\rho,$$

$\epsilon(\rho^{-1}) = \epsilon(\rho)$, $\rho \in \mathbb{R}_+^*$, $\epsilon(\rho)$, for $\rho \geq 1$, in terms of Prolate Wave functions

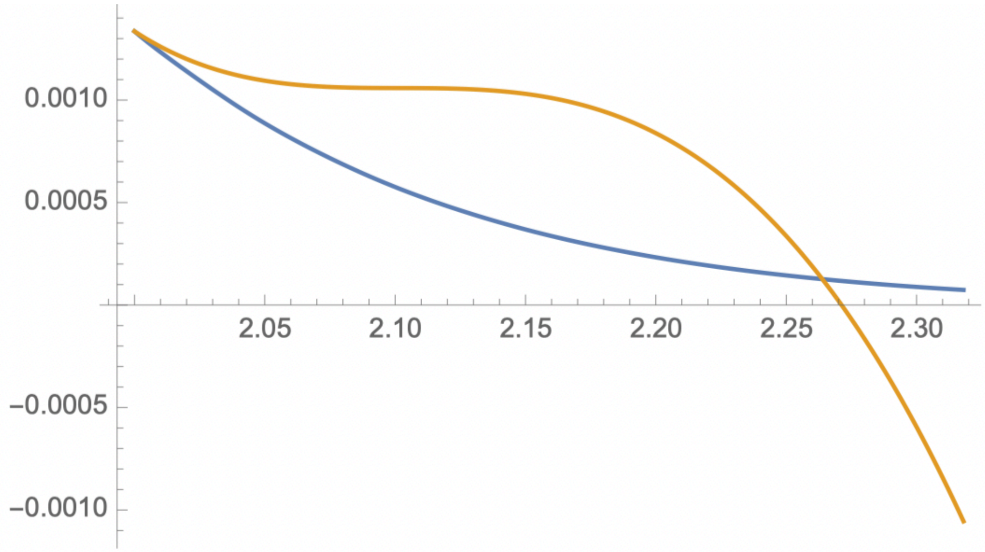
$$\epsilon(\rho) = \sum \frac{\lambda(n)}{\sqrt{1-\lambda(n)^2}} \langle \xi_n \mid \vartheta(\rho^{-1})\zeta_n \rangle.$$

Strong form of Weil positivity (ac+cc)

$$-W_{\mathbb{R}}(f * f^*) \geq \mathrm{Tr}(\vartheta(f)\, S\, \vartheta(f)^*)$$

$$\forall f \in C_c^\infty(\mathbb{R}_+^*),\ \mathrm{support}(f) \subset [2^{-1/2}, 2^{1/2}]$$

$$\widehat{f}\left(\frac{i}{2}\right) = 0,\ \widehat{f}(0) = 0$$



Change of sign of the smallest eigenvalue for the archimedean contribution alone, as a function of $\mu := \exp L$, near $\mu = 2$ (in yellow). After adding the contribution of the prime 2 the smallest eigenvalue of the even matrix is > 0 (in blue)

Minuscule eigenvalues

▶ For $\lambda^2 = 11$ the smallest positive eigenvalue is 2.389×10^{-48}

▶ The presence of these minuscule positive eigenvalues is explained conceptually by the fact that the radical of

Weil's quadratic form contains the range of the map $\mathcal{E}$, for $f(0) = \widehat{f}(0) = 0$

$$\mathcal{E}(f)(x) = x^{1/2} \sum_{n>0} f(nx)$$

- ▶ $f \in P_\lambda \Rightarrow$ support of $\mathcal{E}(f)$ is contained in $(0, \lambda]$
- ▶ $f \in \widehat{P}_\lambda \Rightarrow$ support of $\mathcal{E}(f)$ is contained in $[\lambda^{-1}, \infty)$

Prolate projection $\Pi(\lambda, k)$

$$\mathcal{E}(f)(x) = x^{1/2} \sum_{n>0} f(nx)$$

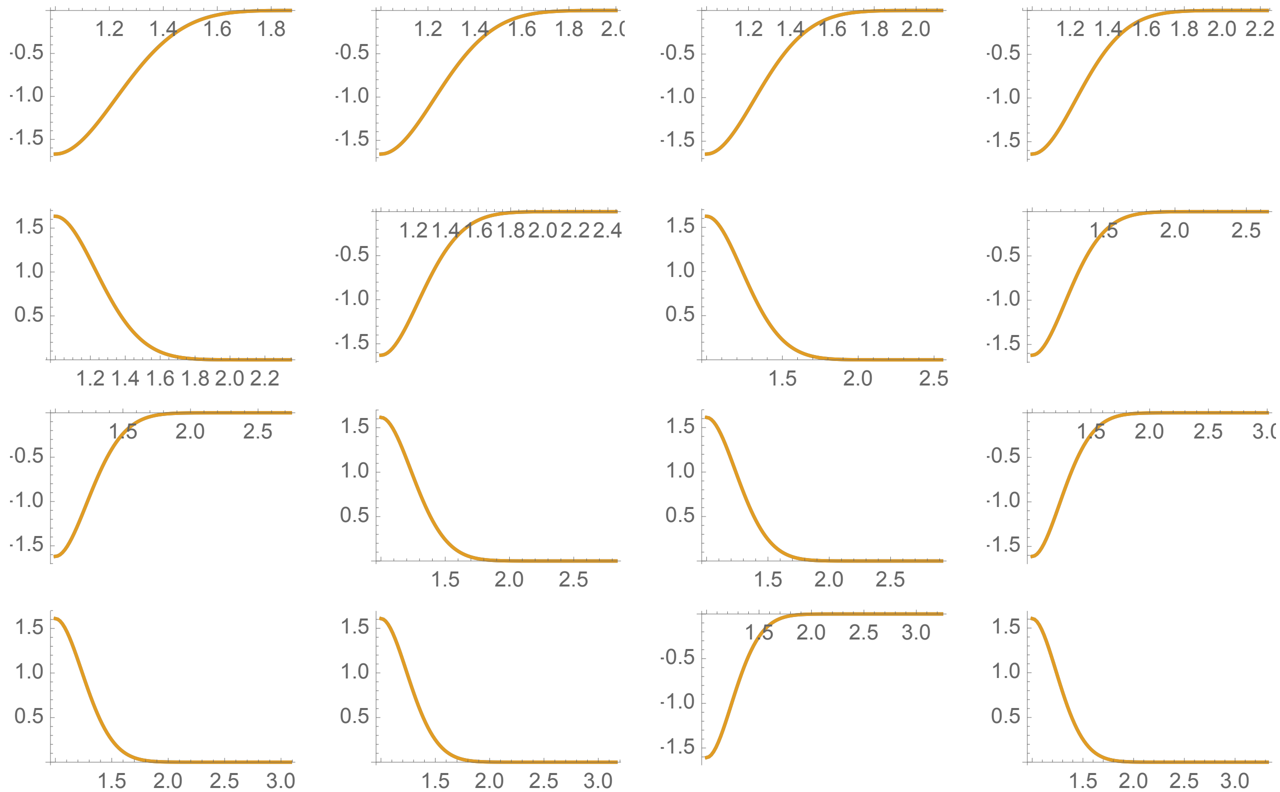
Riemann-Roch = Poisson Formula

$$f(0) = \widehat{f}(0) = 0 \Rightarrow \mathcal{E}(\widehat{f})(x) = \mathcal{E}(f)(x^{-1})$$

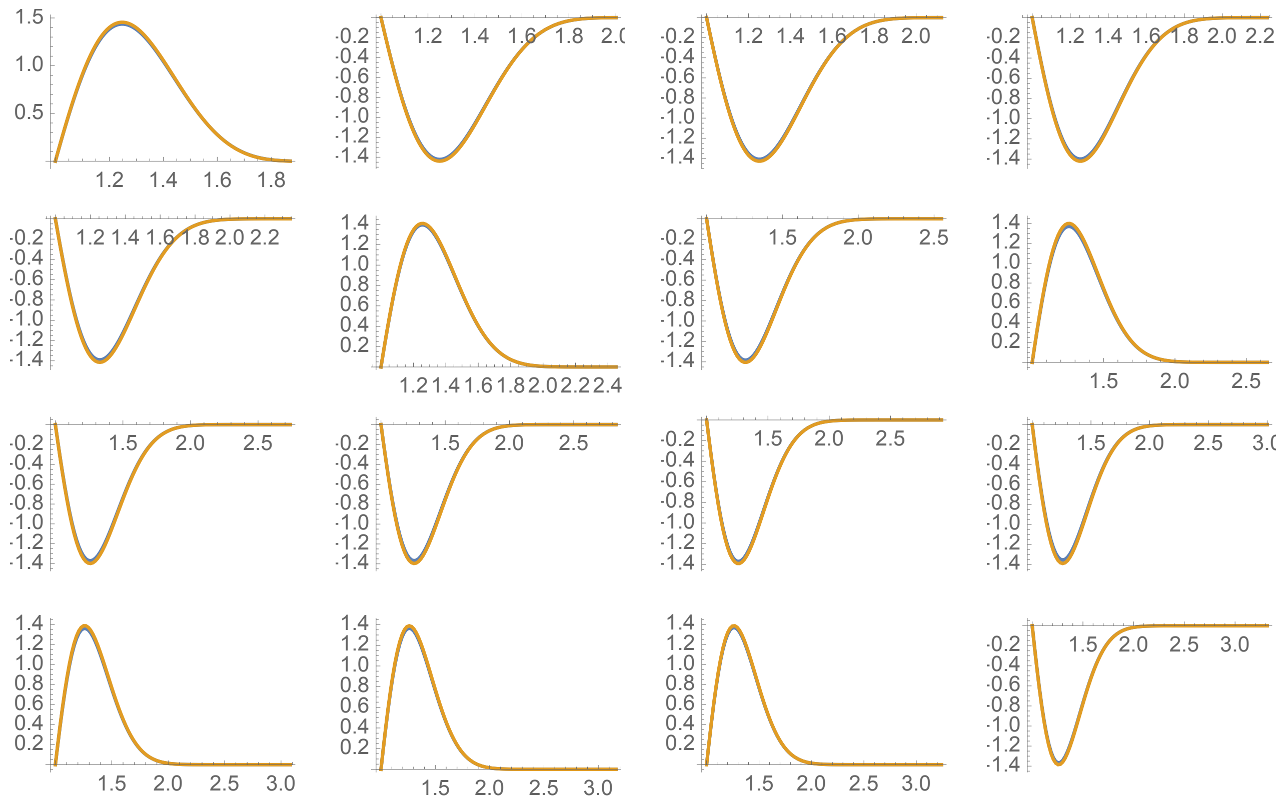
$\Pi(\lambda, k)$ orth. proj. on $\mathcal{E}(E(\lambda, k))$

Fundamental Fact

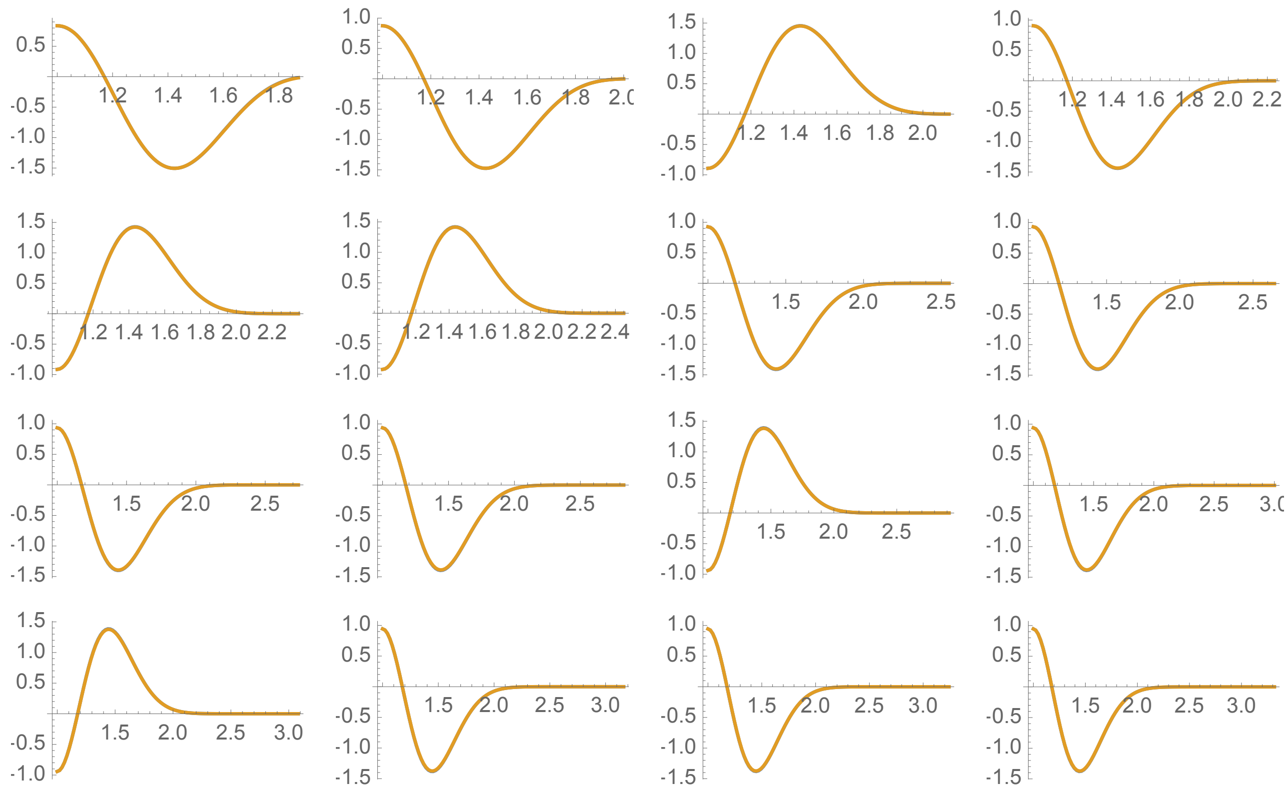
The space of eigenvectors of k lowest eigenvalues for the Weil quadratic form QW_λ corresponds to the prolate projection $\Pi(\lambda, k)$!



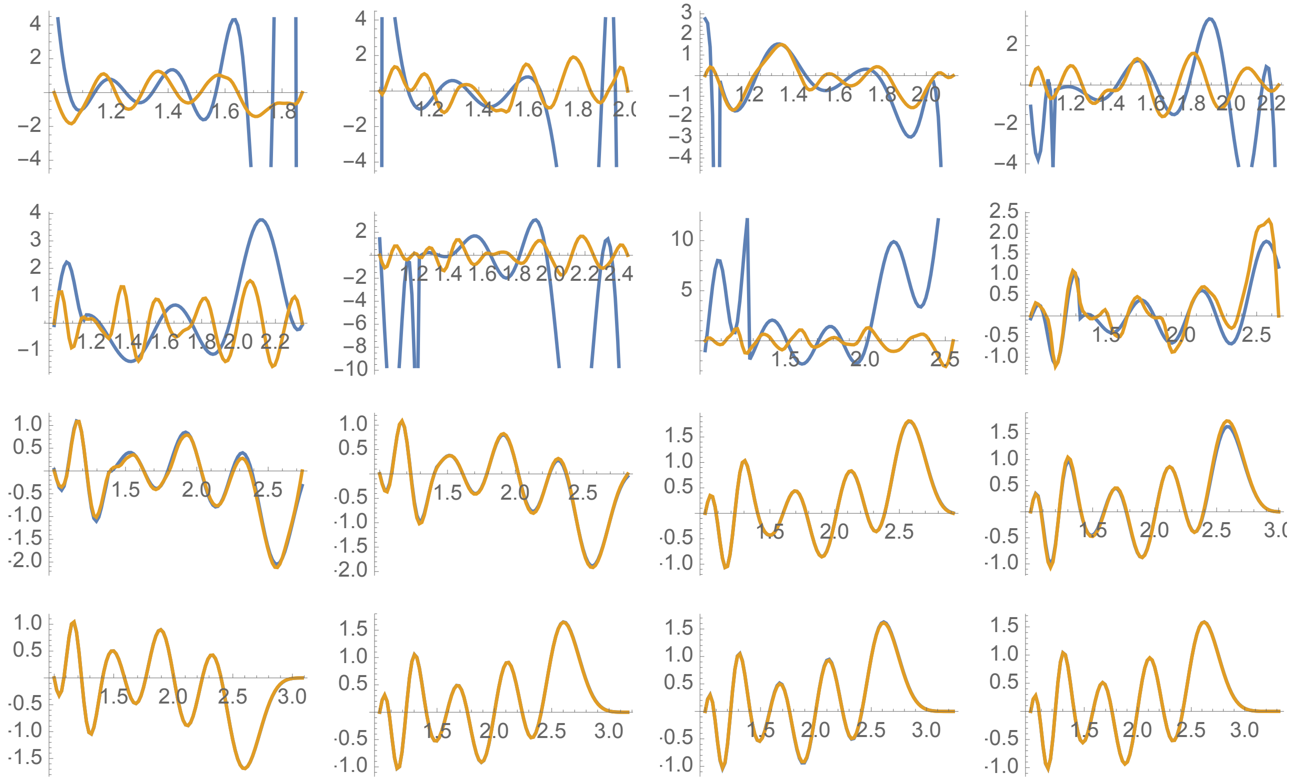
agreement of eigenfunctions for the even matrix and the smallest eigenvalue, for the 16 values of μ between 3.5 and 11. For $\mu = 11$ the eigenvalue is 2.389×10^{-48}



agreement of eigenfunctions for the odd matrix and the smallest eigenvalue for the 16 values of μ between 3.5 and 11



agreement of eigenfunctions for the even matrix and the second smallest eigenvalue for the 16 values of μ between 3.5 and 11



agreement of eigenfunctions for the odd matrix and the 6-th smallest eigenvalue for the 16 values of μ between 3.5 and 11. They begin to agree around $\mu = 7.5$

Eigenvectors described using prolate functions

Use the projection $\Pi(\lambda, k)$ to condition the scaling operator $\rightarrow$ spectral triple

Spectral triple $\Theta(\lambda, k) = (\mathcal{A}(\lambda), \mathcal{H}(\lambda), D(\lambda, k))$

- $\mathcal{A}(\lambda) := C^\infty(\mathbb{R}_+^*/\mu^{\mathbb{Z}})$, $\mu = \lambda^2$

- $\mathcal{H} = L^2(\mathbb{R}_+^*/\mu^{\mathbb{Z}}, d^*u) \simeq L^2([\lambda^{-1}, \lambda], d^*u)$

- $D(\lambda, k) := Q \circ D_0(\lambda) \circ Q$,

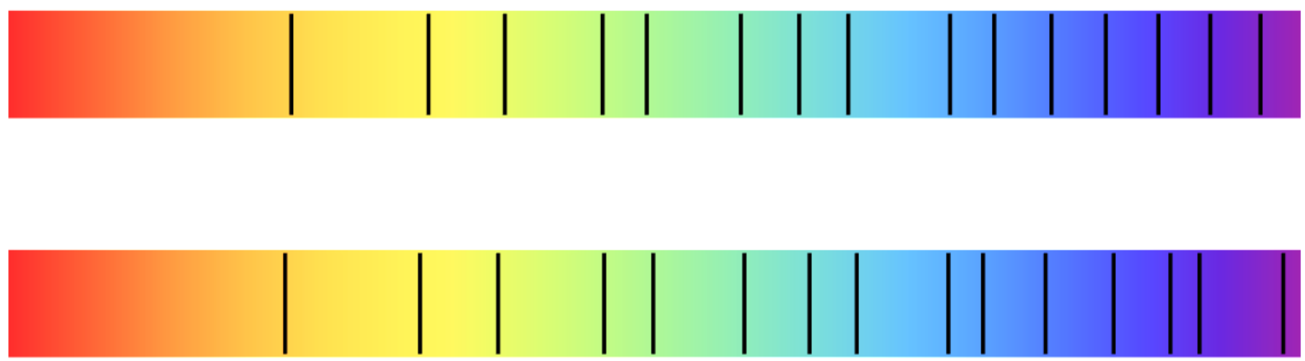
$Q = 1 - \Pi(\lambda, k)$, $D_0(\lambda) := -iu\partial_u$

$\mu = 10.5$

n	$\chi(10.5, n)$
15	0.9999999974270022369
16	0.99999997703659571104
17	0.99999843436641476606
18	0.99992039045021729410
19	0.99713907784499135361
20	0.94005235637340584775
21	0.58413979804862029634
22	0.16939519615152177689

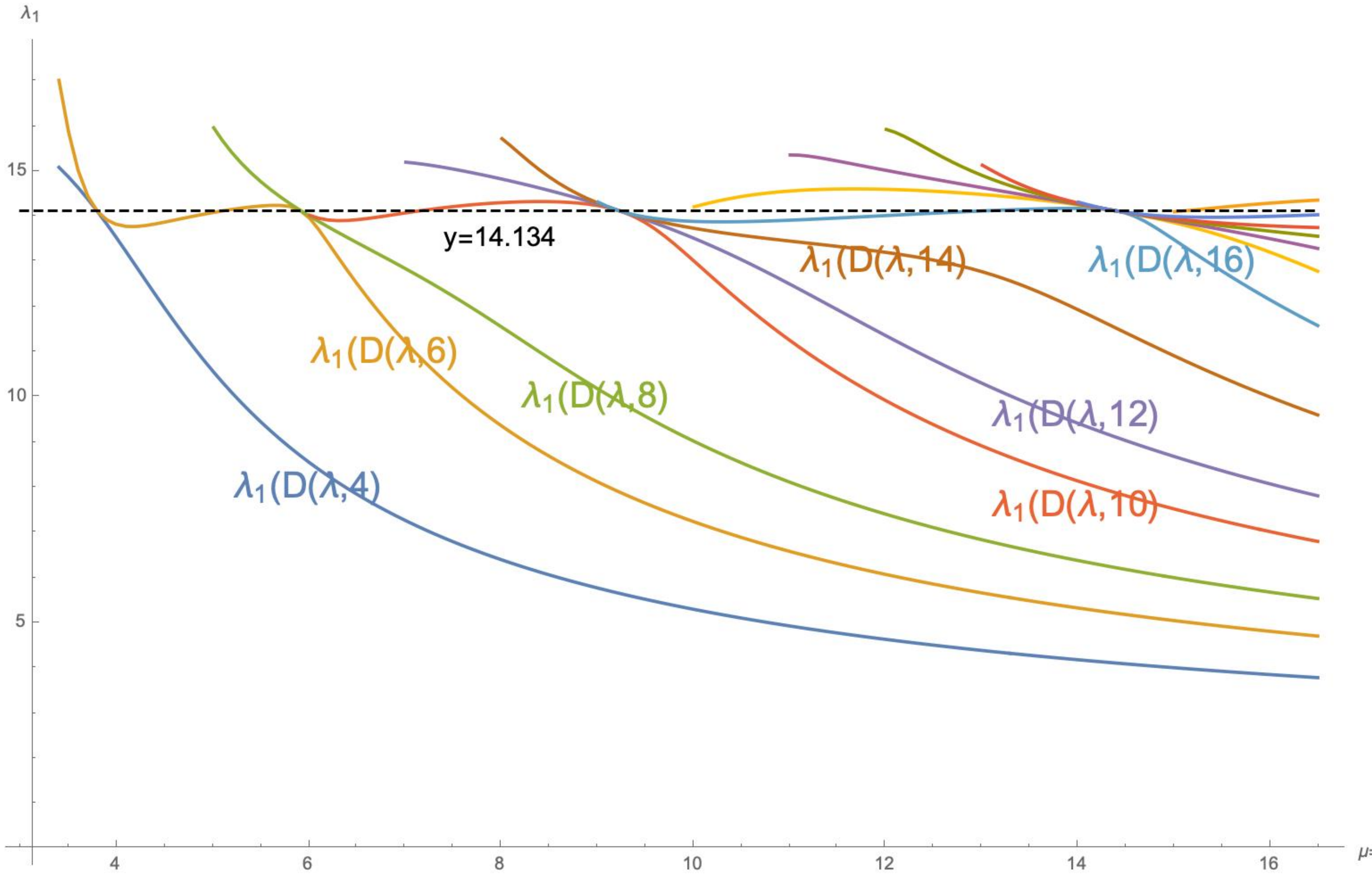
One has $\nu(10.5) = 20$, $2\pi 10.5 \sim 65.9734$.

λ_j	ζ_j
14.450	14.1347
21.455	21.022
25.356	25.0109
30.345	30.4249
32.600	32.9351
37.410	37.5862
40.387	40.9187
42.895	43.3271
48.095	48.0052
50.346	49.7738
53.272	52.9703
56.050	56.4462
58.737	59.347
61.386	60.8318
63.949	65.1125



Evolution of eigenvalues

The curves represent, as a function of $\mu = \lambda^2$, the first positive eigenvalue $\lambda_1(D(\lambda, 2k))$ of $D(\lambda, 2k)$. The ordinate of the points where the graphs touch each other is constant and coincides with the imaginary part $\zeta_1 \sim 14.134$ of the first zero of zeta.



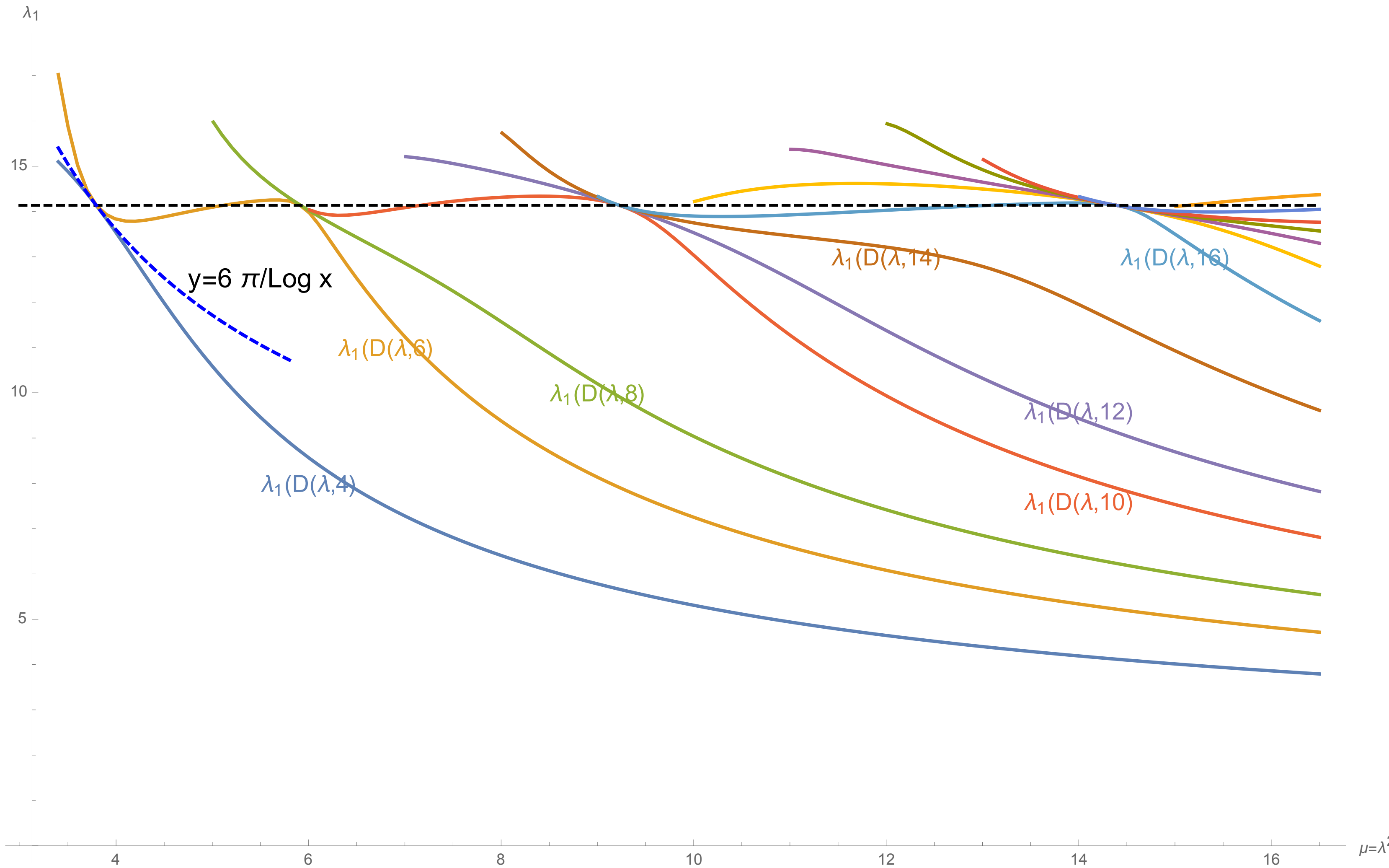
λ_1
15
10
5
y=14.134
$\lambda_1(D(\lambda,4)$
$\lambda_1(D(\lambda,6)$
$\lambda_1(D(\lambda,8)$
$\lambda_1(D(\lambda,10)$
$\lambda_1(D(\lambda,12)$
$\lambda_1(D(\lambda,14)$
$\lambda_1(D(\lambda,16)$
4
6
8
10
12
14
16
$\mu=\lambda^2$

Quantization condition

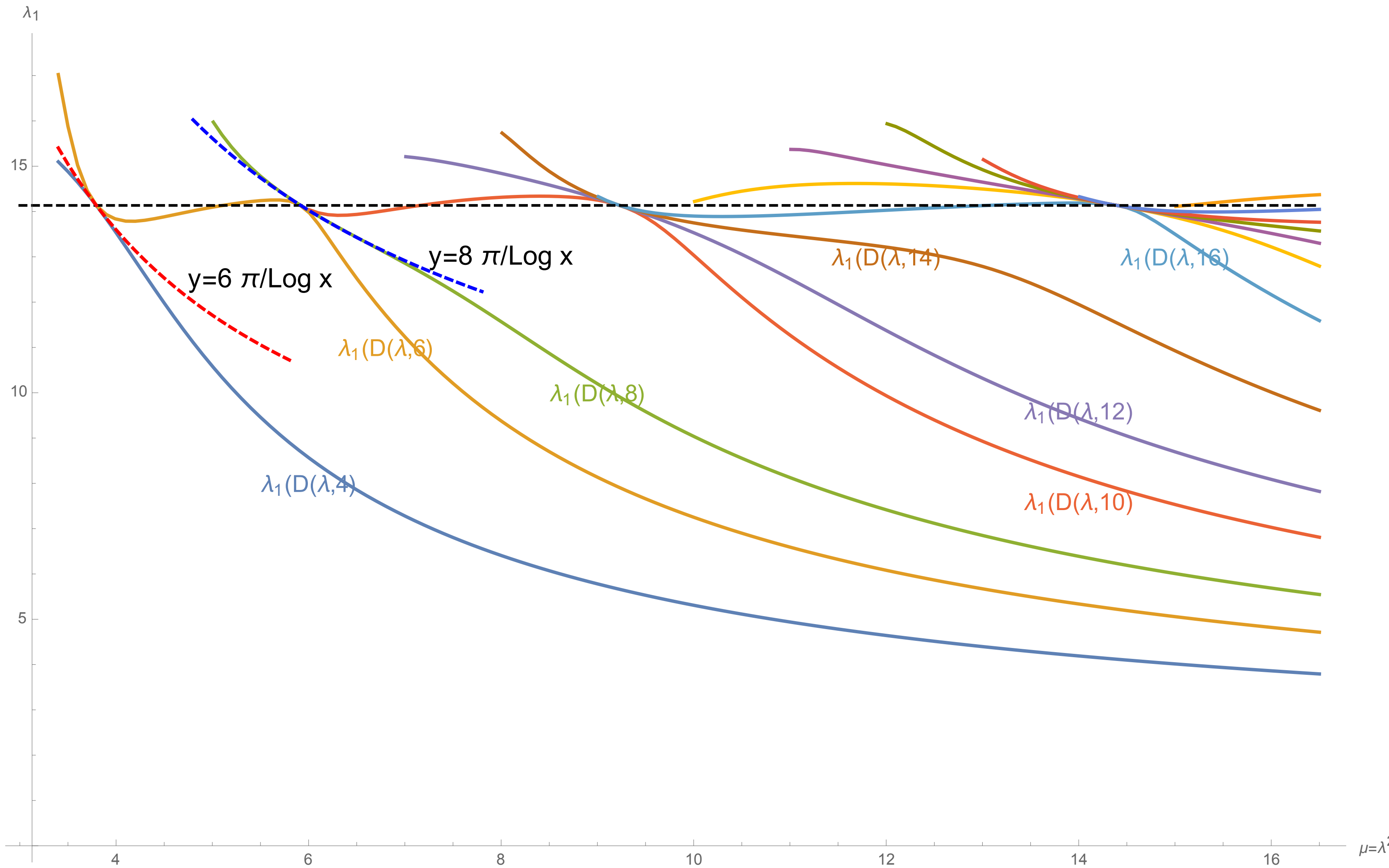
- The points of contact for λ_n fulfill $\log x \in \frac{2\pi}{\zeta_n}\mathbb{Z}$

- The coordinates (x, y) of the points of contact fulfill

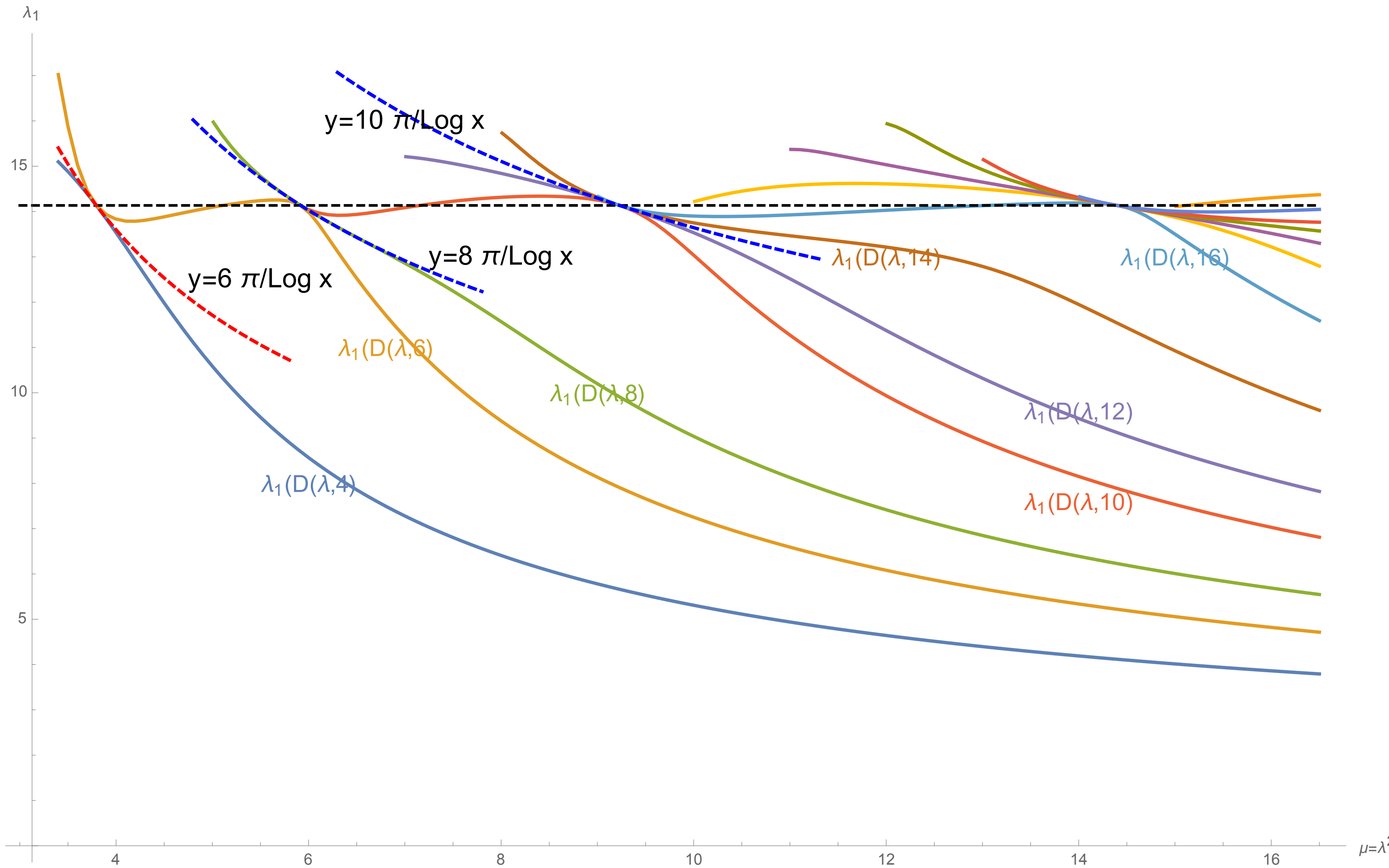
$$x^{iy} = 1$$



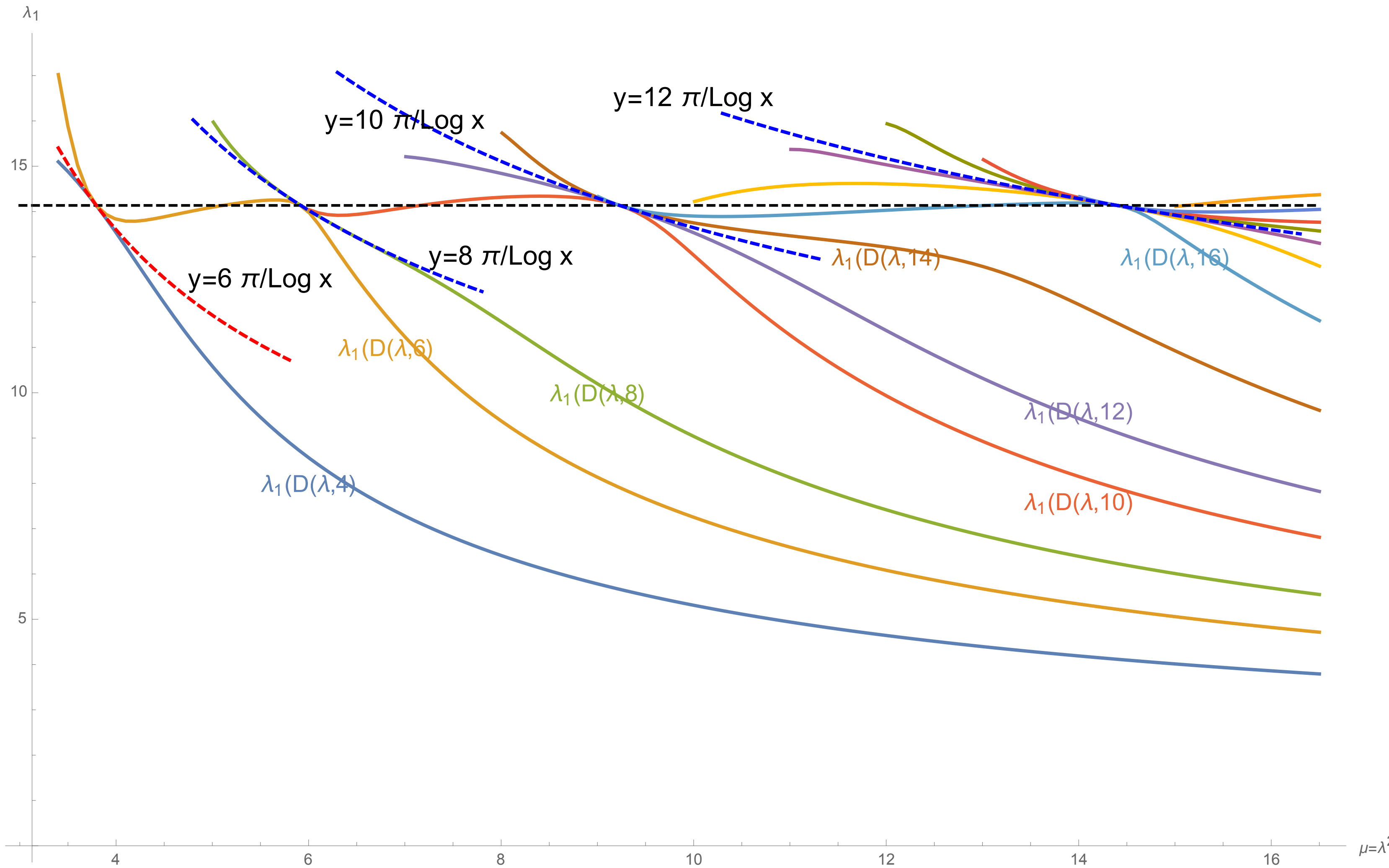
λ_1
15
10
5
4
6
8
10
12
14
16
$\mu=\lambda^2$
y=6 π/Log x
$\lambda_1(D(\lambda,4))$
$\lambda_1(D(\lambda,6))$
$\lambda_1(D(\lambda,8))$
$\lambda_1(D(\lambda,10))$
$\lambda_1(D(\lambda,12))$
$\lambda_1(D(\lambda,14))$
$\lambda_1(D(\lambda,16))$



λ_1
15
10
5
4
6
8
10
12
14
16
$\mu=\lambda^2$
y=6 π/Log x
y=8 π/Log x
λ_1(D(λ,4))
λ_1(D(λ,6))
λ_1(D(λ,8))
λ_1(D(λ,10))
λ_1(D(λ,12))
λ_1(D(λ,14))
λ_1(D(λ,16))

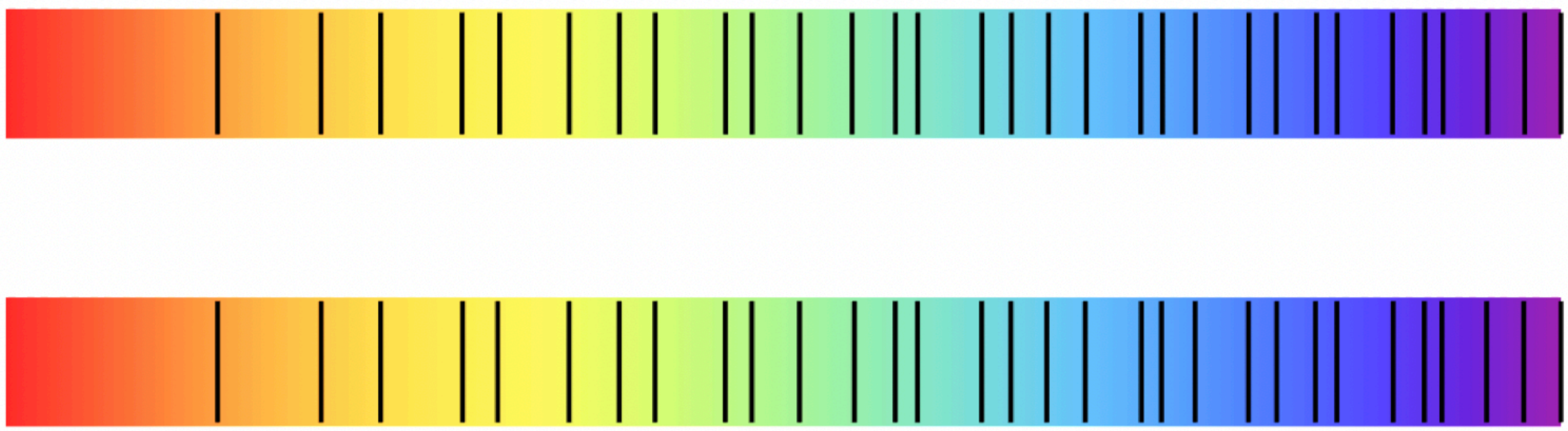


λ_1
15
10
5
4
6
8
10
12
14
16
$\mu=\lambda^2$
y=10 π/Log x
y=8 π/Log x
y=6 π/Log x
λ_1(D(λ,4)
λ_1(D(λ,6)
λ_1(D(λ,8)
λ_1(D(λ,10)
λ_1(D(λ,12)
λ_1(D(λ,14)
λ_1(D(λ,16)



λ_1
15
10
5
4
6
8
10
12
14
16
$\mu=\lambda^2$
y=6 π/Log x
y=8 π/Log x
y=10 π/Log x
y=12 π/Log x
λ_1(D(λ,4))
λ_1(D(λ,6))
λ_1(D(λ,8))
λ_1(D(λ,10))
λ_1(D(λ,12))
λ_1(D(λ,14))
λ_1(D(λ,16))

Criterion $\xi_n(D)$ eigenvector of D_0



Conceptual explanation

- Riemann sums in integration

- Scale invariant Riemann sums

- Zeta-cycles

Scale invariant Riemann sums

$$\int f(u)du = 0, \;\; (f(0) = 0)$$

$$(\mathcal{E}f)(u) := u^{1/2} \sum_{n>0} f(nu)$$

$$(\Sigma_\mu g)(u) := \sum_{k \in \mathbb{Z}} g(\mu^k u).$$

Zeta-cycles

A ζ-**cycle** is a circle C of length

$L = \log \mu$ such that the subspace $\Sigma_\mu \mathcal{E}(\mathcal{S}_0)$

is not dense in the Hilbert space $L^2(C)$.

Theorem

(i) **Let** C **be a** ζ**-cycle. Then the spectrum of the action of the multiplicative group** $\mathbb{R}_+^*$ **on the orthogonal complement of** $\Sigma_\mu \mathcal{E}(\mathcal{S}_0)$ **in** $L^2(C)$ **is formed by imaginary parts of zeros of zeta on the critical line.**

Conversely :

(ii) **Let** $s > 0$ **be such that** $\zeta(\frac{1}{2} + is) = 0$**, then any circle** C **of length an integral multiple of** $2\pi/s$ **is a zeta cycle and its spectrum, for the action of** $\mathbb{R}_+^*$ **on** $\Sigma_\mu \mathcal{E}(\mathcal{S}_0) \subset L^2(C)$**, contains** is

This Theorem provides the theoretical explanation for the above coincidence of spectral values. Indeed, the special values of $\lambda^2 = \mu = \exp L$ at which the k dependence of the eigenvalue $\lambda_n(D(\lambda, k))$ disappear, signal that the related circle of length L is a ζ-cycle and that $\lambda_n(D(\lambda, k))$ is in its spectrum. This explains why the low lying part of the spectrum of the spectral triple $\Theta(\lambda, k)$ possesses a tantalizing resemblance with the low lying zeros of the Riemann zeta function.

Zeta-cycles $\sim$ closed geodesics !

$C = \zeta$-cycle then for any integer $n > 0$

The n-fold cover of C is a ζ-cycle

The length of the ζ-cycles are $\frac{2\pi n}{\zeta_k}$.

Ultraviolet behavior

with H. Moscovici

Prolate Wave Operator

and Zeta

$$N(E) = \frac{E}{2\pi} \log \frac{E}{2\pi} - \frac{E}{2\pi}$$

$$+O(\log E)$$

$$\mathrm{Tr}(\exp(-tD^2)) = \frac{\log\left(\frac{1}{t}\right)}{4\sqrt{\pi}\sqrt{t}} - \frac{(\log 4\pi + \frac{1}{2}\gamma)}{2\sqrt{\pi}\sqrt{t}}$$

$$+O\left(\log\left(\frac{1}{t}\right)\right)$$

A. Connes, H. Moscovici, *Prolate spheroidal operator and Zeta*. arXiv :2112.05500, math.NT math.CA math.QA

A. Connes, H. Moscovici, *The UV prolate spectrum matches the zeros of zeta*. Proc. Natl. Acad. Sci. U.S.A. 119, e2123174119 (2022).

Scaling H does not

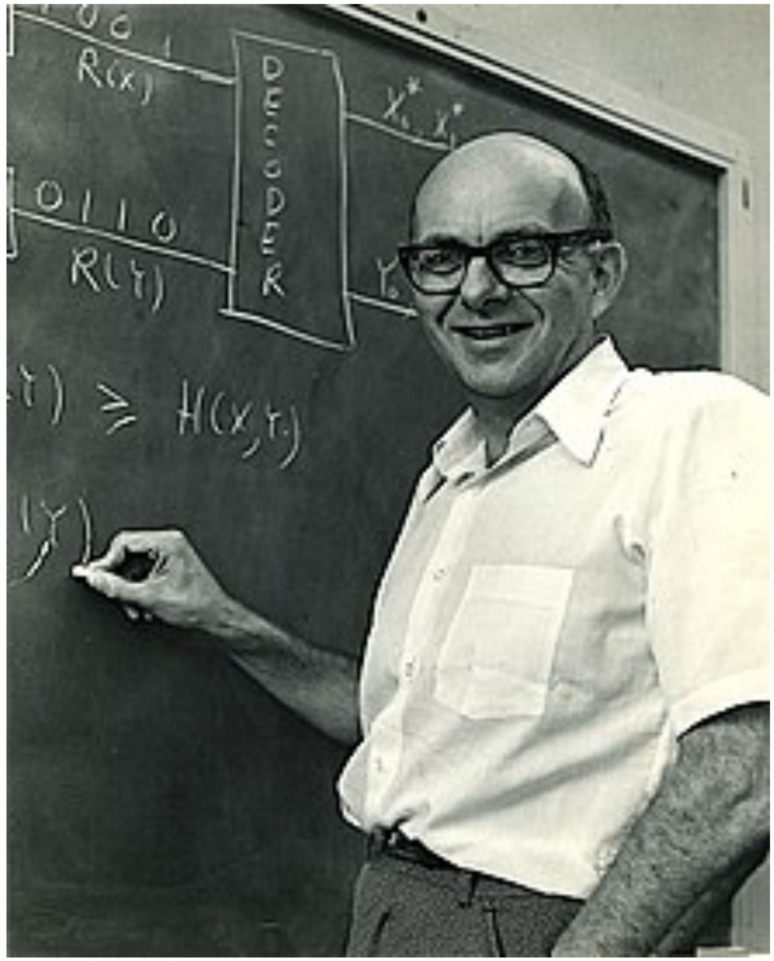
commute with Sonin S_λ

but the prolate wave operator

$W_\lambda = H(1+H) + \lambda^2$ **Hermite**

commutes with Sonin S_λ

D. Slepian et all, Bell-labs, 1960-1965



D. Slepian, H. Pollack, *Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty*, The Bell System technical Journal (1961), 43–63.

D. Slepian, *Some asymptotic expansions for prolate spheroidal wave functions*, J. Math. Phys. Vol. **44** (1965), 99–140.

D. Slepian, *Some comments on Fourier analysis, uncertainty and modeling*, Siam Review. Vol. 23 (1983), 379–393.

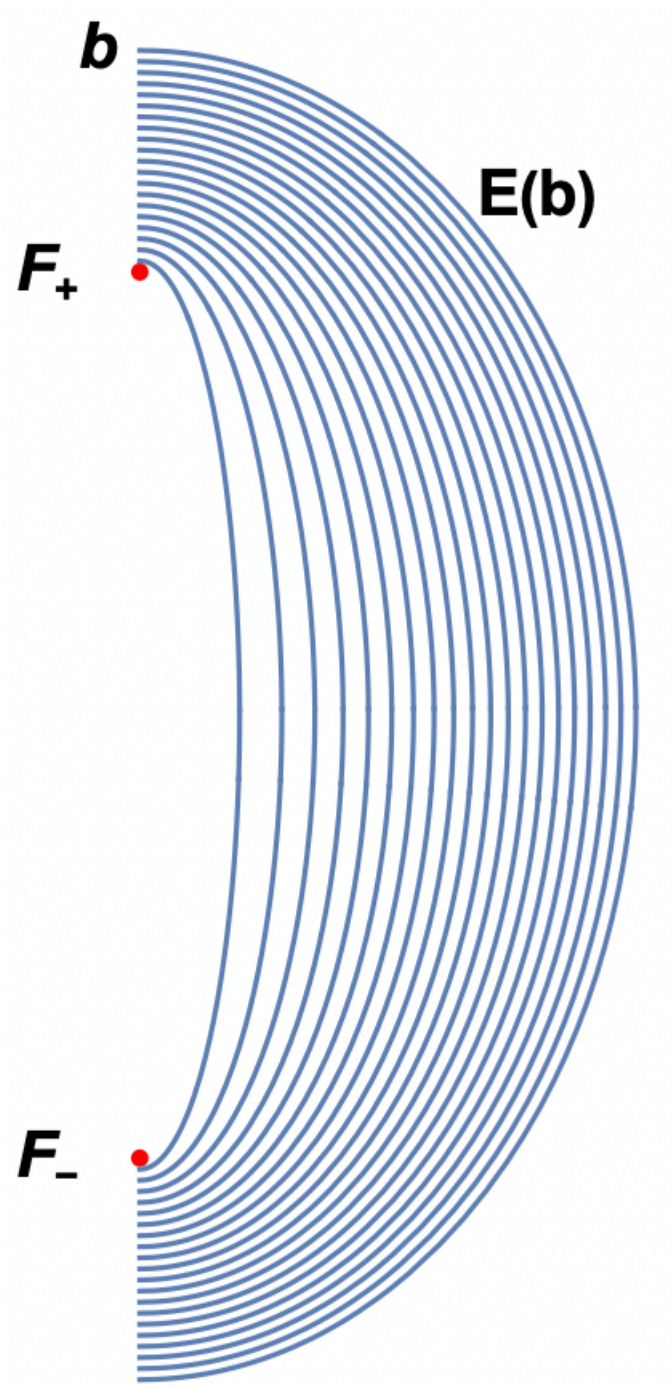
Prolate coordinates

$$x = \sqrt{\left(a^2 - 1\right)\left(1 - b^2\right)}\cos(c)$$

$$y = \sqrt{\left(a^2 - 1\right)\left(1 - b^2\right)}\sin(c)$$

$$z = ab$$

confocal ellipses $E(b)$, focal distance 2, sum of distances $= 2b$



b
E(b)
F+
F–

Helmholtz equation $\Delta + k^2 = 0$

Rotation invariant solutions $\partial_c = 0$

$$\Delta = (a^2 - b^2)^{-1} \left(\partial_a (a^2 - 1) \partial_a + \partial_b (1 - b^2) \partial_b \right)$$

$$+ (a^2 - 1)^{-1} (1 - b^2)^{-1} \partial_c^2$$

$$(a^2 - b^2)(\Delta + k^2) = \partial_a (a^2 - 1) \partial_a + \partial_b (1 - b^2) \partial_b$$

$$+ k^2 (a^2 - b^2)$$

Prolate spheroidal operator

The second order operator $\mathbf{W}_\lambda$ appears from separation of variables in the Laplacian Δ for the prolate spheroid :

$$\mathbf{W}_\lambda := -\partial_x((\lambda^2 - x^2)\partial_x) + (2\pi\lambda x)^2$$

$$(k = 2\pi\lambda^2)$$

Commutation with differential operator

- $x\partial_x$ commutes with $1_{[0,\infty]}$
- $(\lambda^2 - x^2)\partial_x$ commutes with $1_{[-\lambda,\lambda]}$
- $\partial_x(\lambda^2 - x^2)\partial_x$ commutes with $1_{[-\lambda,\lambda]}$

Commutation with P_λ and $\widehat{P}_\lambda$

- The operator

$$\mathbf{W}_\lambda := -\partial_x((\lambda^2 - x^2)\partial_x) + (2\pi\lambda x)^2$$

is invariant under $\mathbb{F}_{e_\mathbb{R}}$.

- $\mathbf{W}_\lambda$ commutes with P_λ and $\widehat{P}_\lambda$

Self-adjoint extension

▶ The minimal domain is the Schwartz space $\mathcal{S}(\mathbb{R})$

▶ The deficiency indices are (4,4).

▶ Unique self-adjoint extension W_λ commuting with P_λ and $\widehat{P}_\lambda$.

▶ W_λ commutes with Fourier

▶ The selfadjoint operator W_λ has discrete spectrum.

▶ ϕ eigenfunction of $W_\lambda \Rightarrow$

$$\phi(x) \sim c\,\frac{\sin(2\pi\lambda x)}{x}, \quad x \to \infty$$

if ϕ is even and $\frac{\cos(2\pi\lambda x)}{x}$ if ϕ is odd.

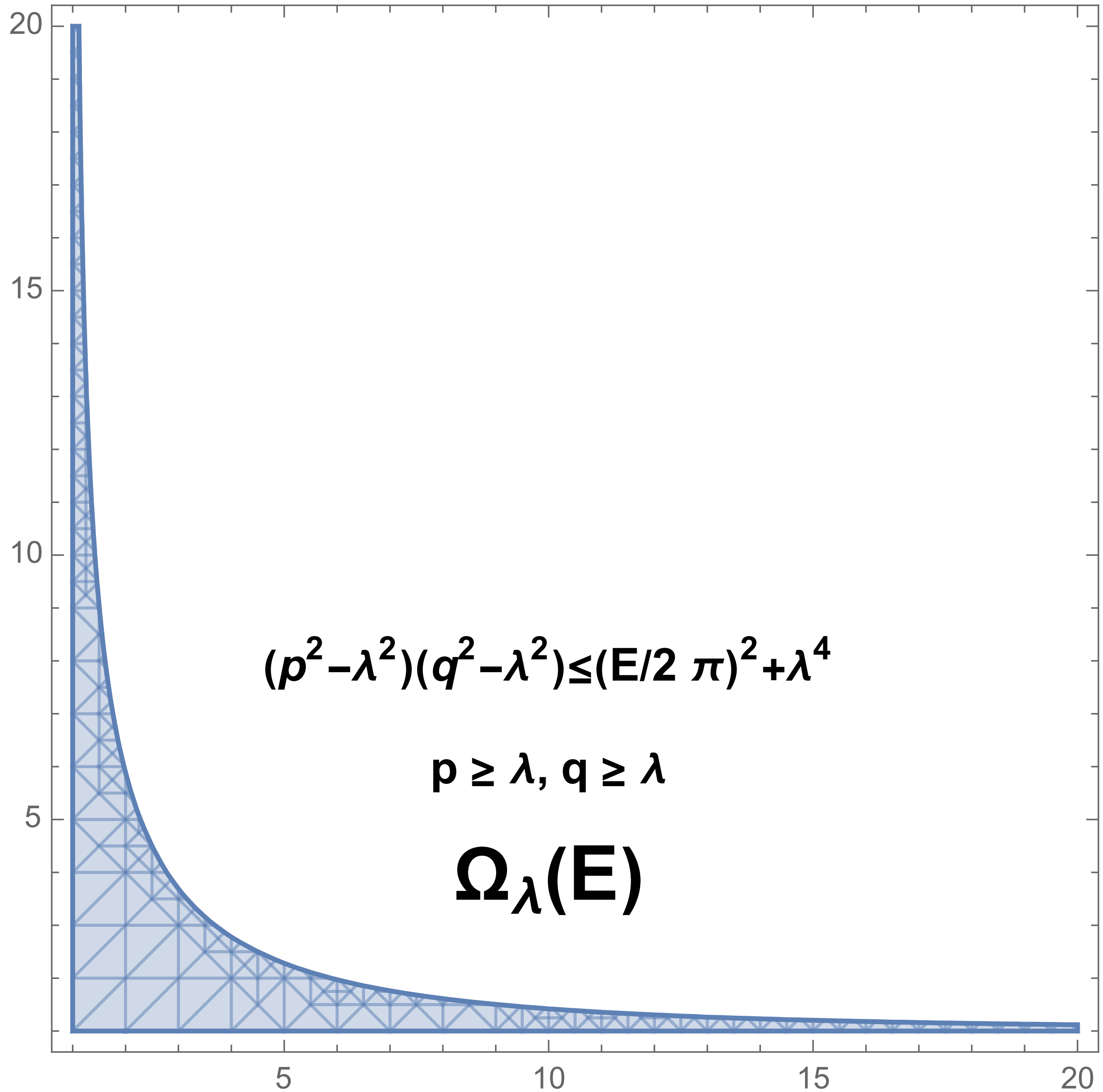
Semiclassical approximation

$$H_\lambda(p,q) = (p^2 - \lambda^2)(q^2 - \lambda^2)$$

$$W_\lambda = -4\pi^2 H_\lambda + 4\pi^2 \lambda^4$$

$$\Omega_\lambda(E) := \{(q,p) \mid q \geq \lambda, p \geq \lambda, H_\lambda(p,q) \leq a\}$$

$$a = \left(\frac{E}{2\pi}\right)^2$$


20
15
10
5
5
10
15
20
$(p^2-\lambda^2)(q^2-\lambda^2)\leq(E/2\pi)^2+\lambda^4$
$p \geq \lambda$, $q \geq \lambda$
$\Omega_\lambda(E)$

The area $\sigma(E)$ of $\Omega_\lambda(E)$ is given, with $a = \left(\frac{E}{2\pi}\right)^2$, by the convergent integral

$$I_\lambda(a) = \int_\lambda^\infty \left(\frac{\sqrt{a + \lambda^2 x^2 - \lambda^4}}{\sqrt{x^2 - \lambda^2}} - \lambda \right) dx$$

$$\sigma(E) \sim \frac{E}{2\pi}\left(\log\left(\frac{E}{2\pi}\right) - 1 + \log(4) - 2\log(\lambda) \right)$$

$$+\lambda^2 + o(1)$$

In fact one has

$$I_\lambda(a) = \lambda^2 I_1(a\,\lambda^{-4})$$

and in terms of elliptic integrals

$$I_1(a) = aK(1-a) - E(1-a) + 1$$

$$\sim \frac{1}{2}\sqrt{a}(\log(a) - 2 + 2\log(4)) + 1 + o(1)$$

Liouville transform

$$V(f)(y) := \Lambda^{1/2} f(\Lambda \cosh(y)) \sinh(y)^{1/2}$$

The operator V is a unitary isomorphism $V : L^2([\Lambda, \infty) \to L^2([0, \infty))$ which conjugates the operator W with the operator

$$S(\phi)(y) := \partial_y^2 \phi(y) - Q(y)\phi(y)$$

$$Q(y) = -(2\pi\Lambda^2)^2 \cosh(y)^2 - \frac{1}{4}\left(\coth^2(y) - 2\right)$$

Hamiltonian $H = p^2 + Q(q)$

(i) The Hamiltonian $H = -S$ is in the limit circle case at ∞.
(ii) The Hamiltonian H is in the limit circle case at 0.
Case $\Lambda = \sqrt{2}$ we get for the function $h = -Q$

$$h(y) = 16\pi^2 \cosh^2(y) + \frac{1}{4}\Big(\coth^2(y) - 2\Big)$$

M. Nursultanov, G. Rozenblum, *Eigenvalue asymptotics for the Sturm-Liouville operator with potential having a strong local negative singularity*. Opuscula Mathematica 37(1) :109

$h(p(\mu)) = \mu$, are

$$N(H,(0,\lambda)) = \pi^{-1}\int_0^\infty [(\lambda + h(x))^{\frac{1}{2}} - h(x)^{\frac{1}{2}}]dx + O(1),\ \lambda > 0, \tag{1.4}$$

$$N(H,(-\mu,0)) = \pi^{-1}\int_0^{p(\mu)} h(x)^{\frac{1}{2}}dx + \pi^{-1}\int_{p(\mu)}^\infty [h(x)^{\frac{1}{2}} - (h(x)-\mu)^{\frac{1}{2}}]dx + O(1). \tag{1.5}$$

Formula for $N(a)$

$$N(a) = \frac{1}{\pi}\int_0^\infty \left((a+h(y))^{1/2} - h(y)^{1/2}\right) dy$$

At the level of the Dirac operator one has $a = (E/2)^2$

$$N_D(E) = \frac{E}{2\pi}\log\frac{E}{2\pi} - \frac{E}{2\pi} + O(\log(E))$$

The logarithmic term is $-\frac{1}{2\pi}\log E$. The numerical value of the coefficient is 0.159155 which is of the same order as the constant involved in the estimate of Trudgian for Zeta

$$|N_\zeta(E) - \left(\frac{E}{2\pi}\log\frac{E}{2\pi} - \frac{E}{2\pi}\right)| \leq 0.112\,\log(E) + O(\log\log E)$$

Dirac operator

- We found Dirac operator with Laplacian two copies of W_λ, using the Darboux method.

- We explore associated geometry.

Darboux method

$$p(x) = x^2 - \lambda^2,\ V(x) = 4\pi^2\lambda^2 x^2,\ W_\lambda = \partial(p(x)\partial) + V(x),$$

$$U : L^2([\lambda,\infty), dx) \to L^2([\lambda,\infty), p(x)^{-1/2}dx)$$

$$U(\xi)(x) := p(x)^{1/4}\xi(x), \quad (\delta f)(x) := p(x)^{1/2}\partial f(x)$$

$$\delta w(x) + w(x)^2 = -V(x) + \left(\frac{p''(x)}{4} - \frac{p'(x)^2}{16p(x)}\right), \quad \forall x \in [\lambda,\infty)$$

$$W_\lambda = U^* (\delta + w)(\delta - w) U$$

Solution of Riccati equation

For $z \in \mathbb{C}$ and $u = u_1 + zu_2$ the solution u has no zero in (λ, ∞) if $z \notin \mathbb{R}$ and an infinity of zeros otherwise.

Solutions of the Riccati equation

$$w_z(x) = \frac{(x^2 - \lambda^2)^{1/4} \partial \left((x^2 - \lambda^2)^{1/4} u(x) \right)}{u(x)}$$

where $u = u_1 + zu_2$ and $z \in \mathbb{C} \setminus \mathbb{R}$.

Dirac operator

$$D = \begin{pmatrix} 0 & \delta + w(x) \\ \delta - w(x) & 0 \end{pmatrix}$$

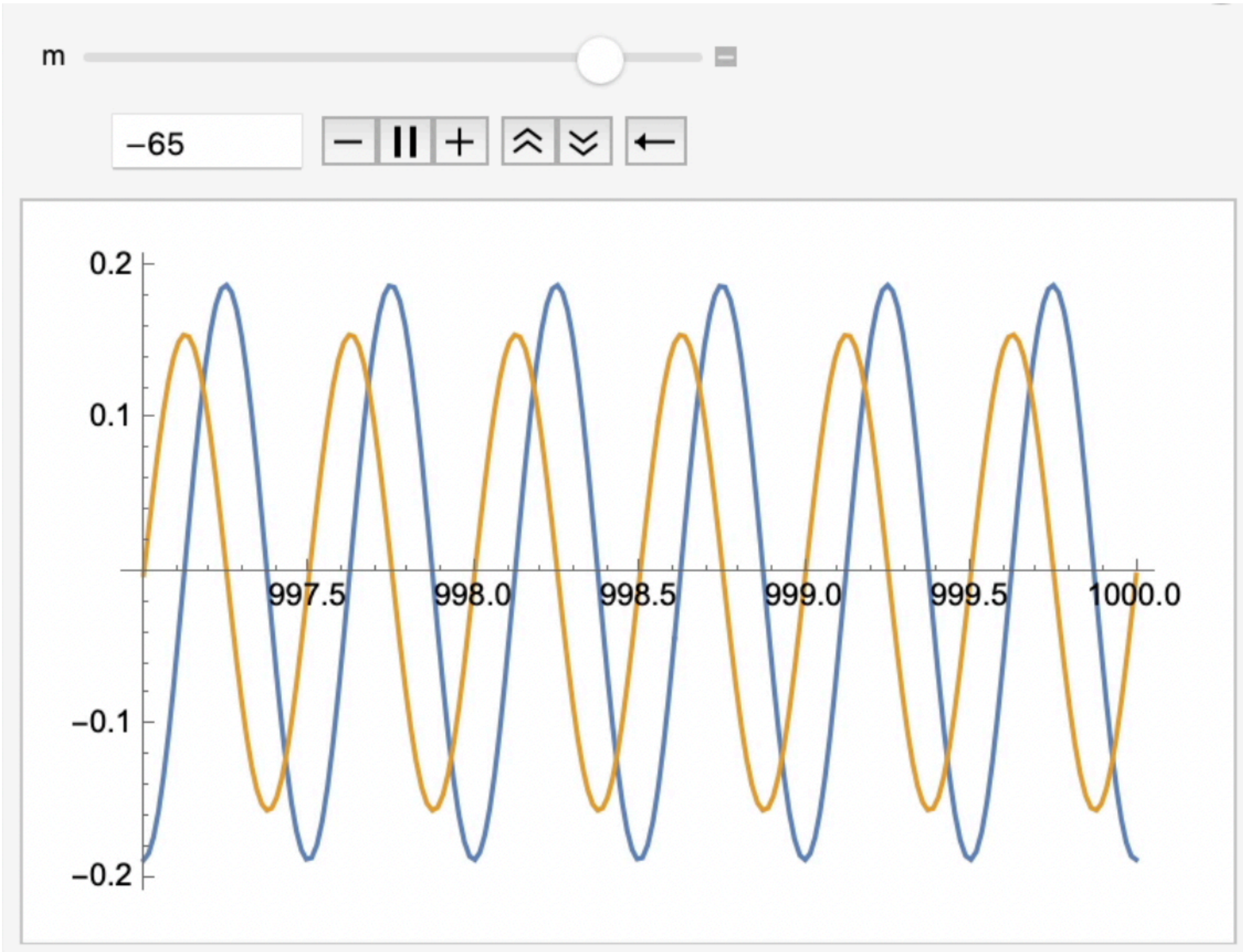
Then the square of D is diagonal with each diagonal term spectrally equivalent to W_λ,

$$U^* D^2 U = \begin{pmatrix} W_\lambda & 0 \\ 0 & W_\lambda + 2\delta w(x) \end{pmatrix}$$

Ultraviolet∼Zeta

The operator $2D$ has discrete simple spectrum contained in $\mathbb{R} \cup i\mathbb{R}$. Its imaginary eigenvalues are symmetric under complex conjugation and the counting function $N(E)$ counting those of positive imaginary part less than E fulfills

$$N(E) \sim \frac{E}{2\pi}\left(\log\left(\frac{E}{2\pi}\right) - 1\right) + O(\log E)$$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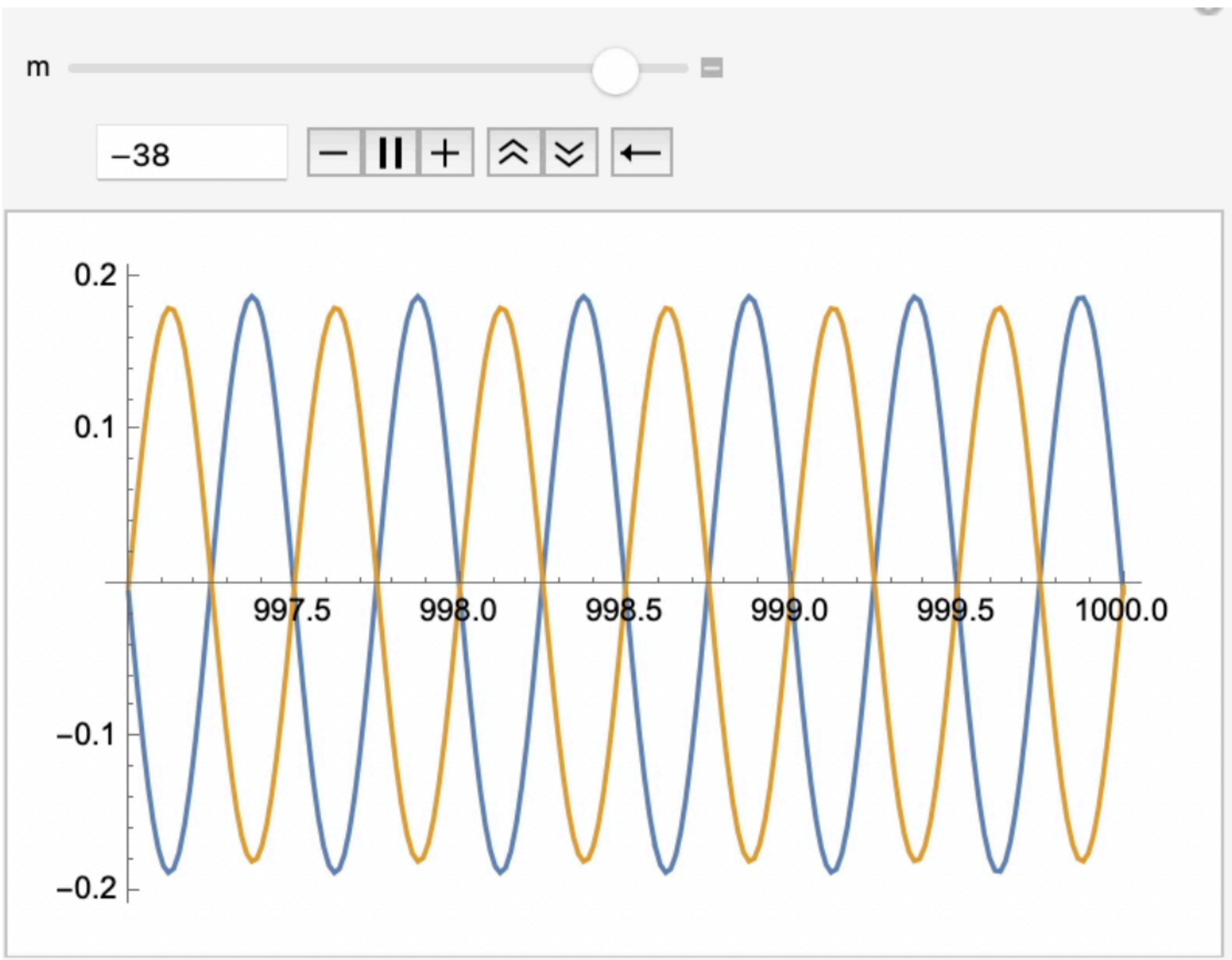
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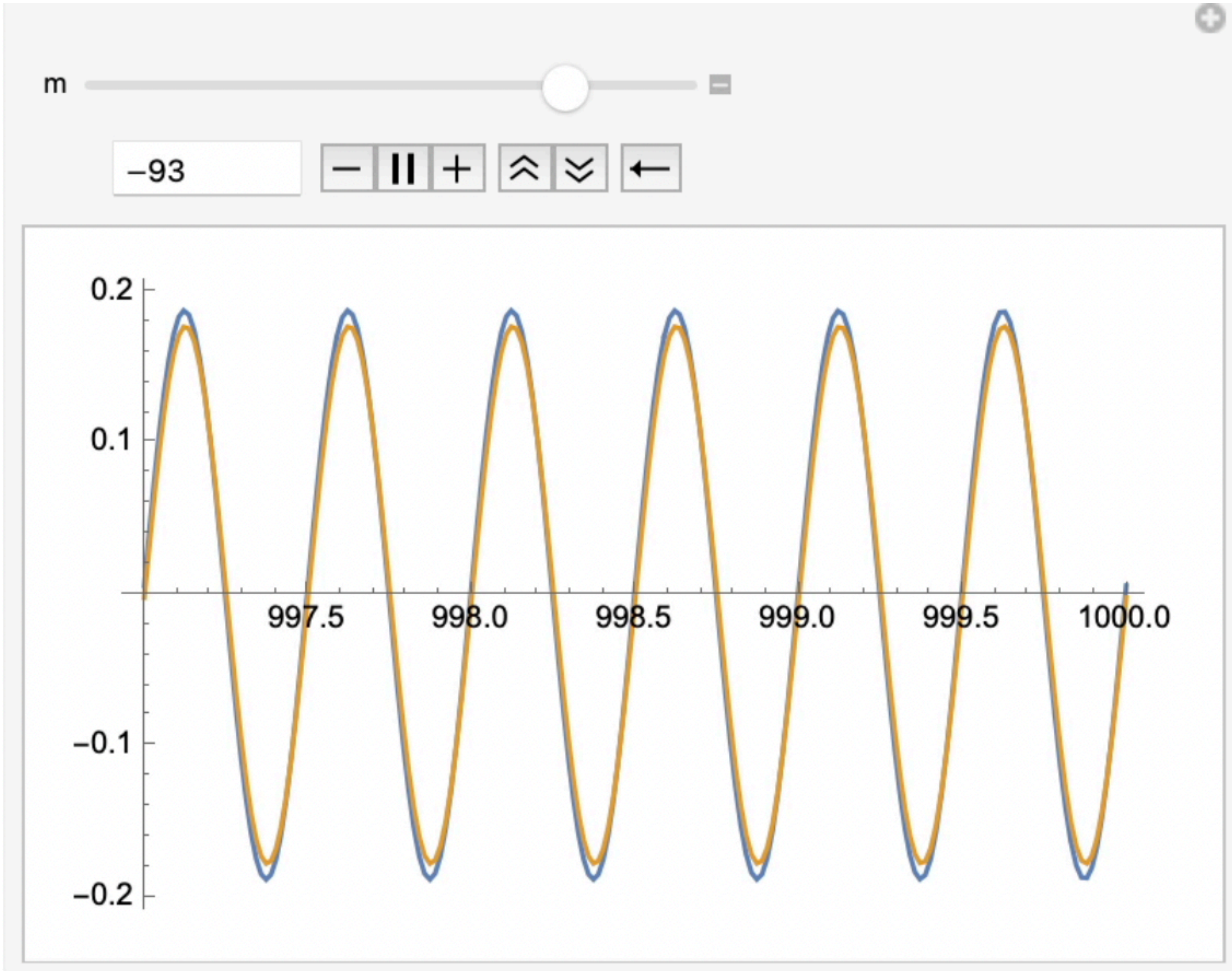

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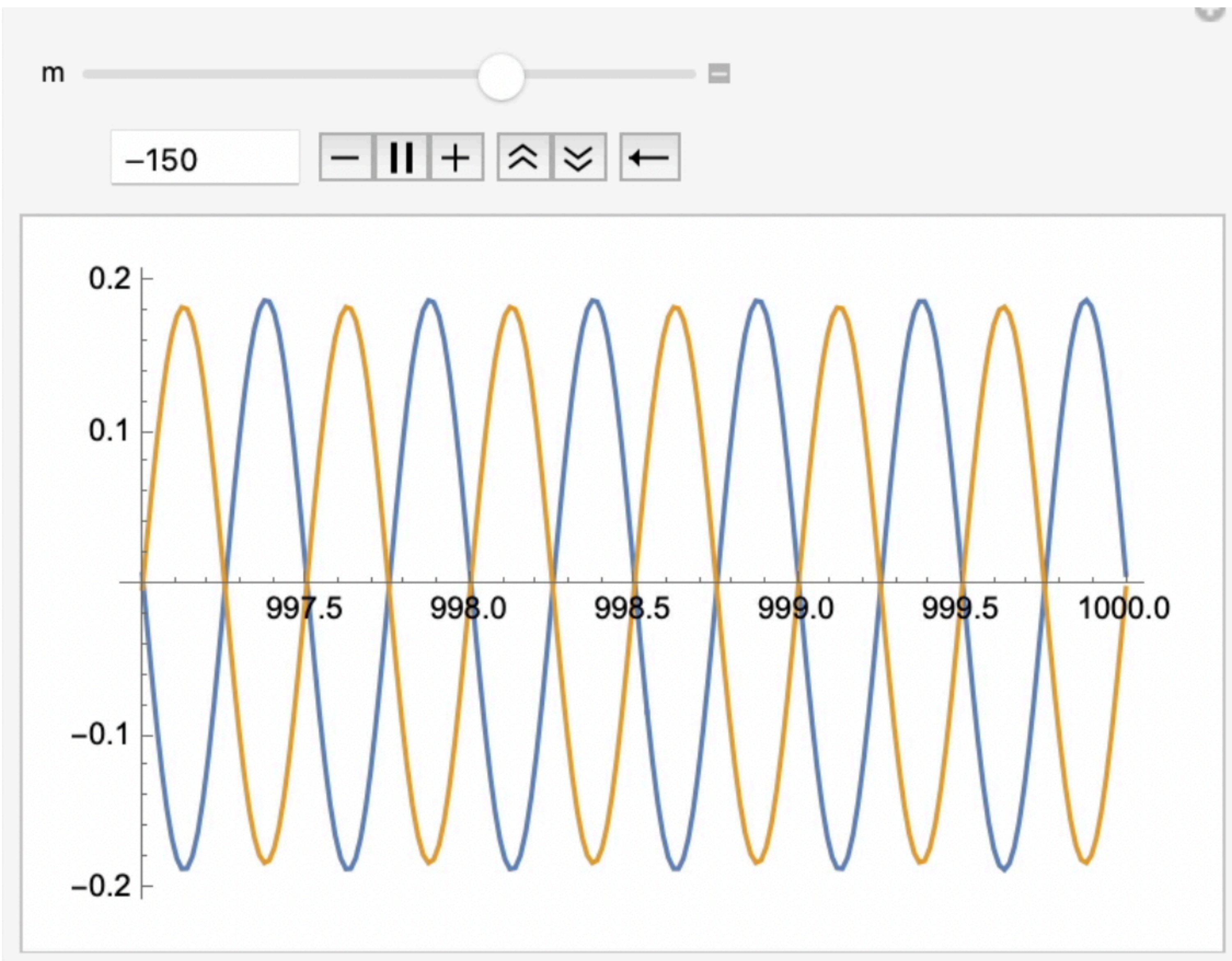
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–0.2
997.5
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998.5
999.0
999.5
1000.0

The first approximate negative eigenvalues of W are

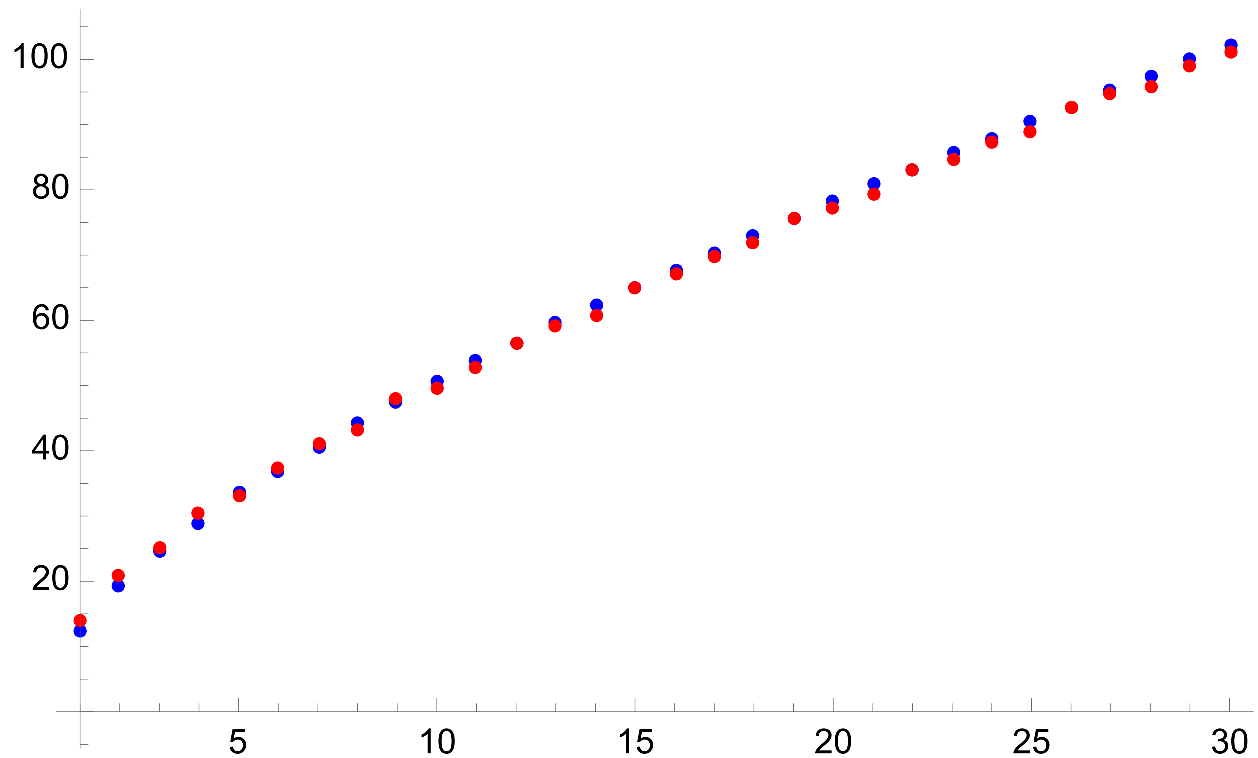
$$-39, -94, -152, -211, -279, -342, -416, -489, -561, -639, -718, -800, -887, -971,$$
$$-1058, -1148, -1242, -1337, -1433, -1528, -1627, -1728, -1834, -1940, -2044, -2155,$$
$$-2262, -2375, -2491, -2606, -2723, -2842, -2964, -3084, -3205, -3330, -3461, -3586,$$
$$-3716, -3845, -3977, -4112, -4245, -4381, -4523, -4662, -4803, -4943, -5088, -5232,$$
$$-5382, -5527, -5677, -5823, -5977, -6129, -6287, -6440, -6600, -6753, -6915, -7075,$$
$$-7240, -7402, -7562, -7730, -7902, -8064, -8237, -8408, -8581, -8748, -8924, -9100,$$
$$-9278, -9456, -9638, -9816, -10000, -10179, -10363, -10549, -10734, -10923, -11114,$$
$$-11299, -11491, -11681, -11876, -12066, -12267, -12459, -12660, -12860, -13059,$$
$$-13254, -13464, -13660, -13865, -14069, -14279, -14484, -14694, -14900, -15113,$$
$$-15326, -15543, -15753, -15967$$

The comparison of $2\sqrt{-z}$ with the zeros of zeta then gives

$$\begin{pmatrix} 12.49 & 14.1347 \\ 19.3907 & 21.022 \\ 24.6577 & 25.0109 \\ 29.0517 & 30.4249 \\ 33.4066 & 32.9351 \\ 36.9865 & 37.5862 \\ 40.7922 & 40.9187 \\ 44.2267 & 43.3271 \\ 47.3709 & 48.0052 \\ 50.5569 & 49.7738 \\ 53.591 & 52.9703 \\ 56.5685 & 56.4462 \\ 59.5651 & 59.347 \\ 62.3217 & 60.8318 \\ 65.0538 & 65.1125 \\ 67.7643 & 67.0798 \\ 70.484 & 69.5464 \\ 73.13 & 72.0672 \\ 75.71 & 75.7047 \\ 78.1793 & 77.1448 \\ 80.6722 & 79.3374 \\ 83.1384 & 82.9104 \\ 85.6505 & 84.7355 \\ 88.0909 & 87.4253 \\ 90.4212 & 88.8091 \\ 92.844 & 92.4919 \\ 95.121 & 94.6513 \\ 97.4679 & 95.8706 \\ 99.8198 & 98.8312 \end{pmatrix}$$

$$\begin{pmatrix} 102.098 & 101.318 \\ 104.365 & 103.726 \\ 106.621 & 105.447 \\ 108.885 & 107.169 \\ 111.068 & 111.03 \\ 113.225 & 111.875 \\ 115.412 & 114.32 \\ 117.661 & 116.227 \\ 119.766 & 118.791 \\ 121.918 & 121.37 \\ 124.016 & 122.947 \\ 126.127 & 124.257 \\ 128.25 & 127.517 \\ 130.307 & 129.579 \\ 132.378 & 131.088 \\ 134.507 & 133.498 \\ 136.558 & 134.757 \\ 138.607 & 138.116 \\ 140.613 & 139.736 \\ 142.66 & 141.124 \\ 144.665 & 143.112 \\ 146.724 & 146.001 \\ 148.688 & 147.423 \\ 150.692 & 150.054 \\ 152.617 & 150.925 \\ 154.622 & 153.025 \\ 156.576 & 156.113 \\ 158.581 & 157.598 \\ 160.499 & 158.85 \\ 162.481 & 161.189 \end{pmatrix}$$

$$\begin{pmatrix} 164.353 & 163.031 \\ 166.313 & 165.537 \\ 168.226 & 167.184 \\ 170.176 & 169.095 \\ 172.07 & 169.912 \\ 173.92 & 173.412 \\ 175.841 & 174.754 \\ 177.786 & 176.441 \\ 179.6 & 178.377 \\ 181.516 & 179.916 \\ 183.39 & 182.207 \\ 185.267 & 184.874 \\ 187.061 & 185.599 \\ 188.934 & 187.229 \\ 190.788 & 189.416 \\ 192.645 & 192.027 \\ 194.484 & 193.08 \\ 196.347 & 195.265 \\ 198.151 & 196.876 \\ 200. & 198.015 \\ 201.782 & 201.265 \\ 203.598 & 202.494 \\ 205.417 & 204.19 \\ 207.21 & 205.395 \\ 209.026 & 207.906 \\ 210.846 & 209.577 \\ 212.594 & 211.691 \\ 214.392 & 213.348 \\ 216.157 & 214.547 \\ 217.954 & 216.17 \\ 219.691 & 219.068 \\ 221.513 & 220.715 \\ 223.24 & 221.431 \\ 225.033 & 224.007 \\ 226.804 & 224.983 \\ 228.552 & 227.421 \end{pmatrix}$$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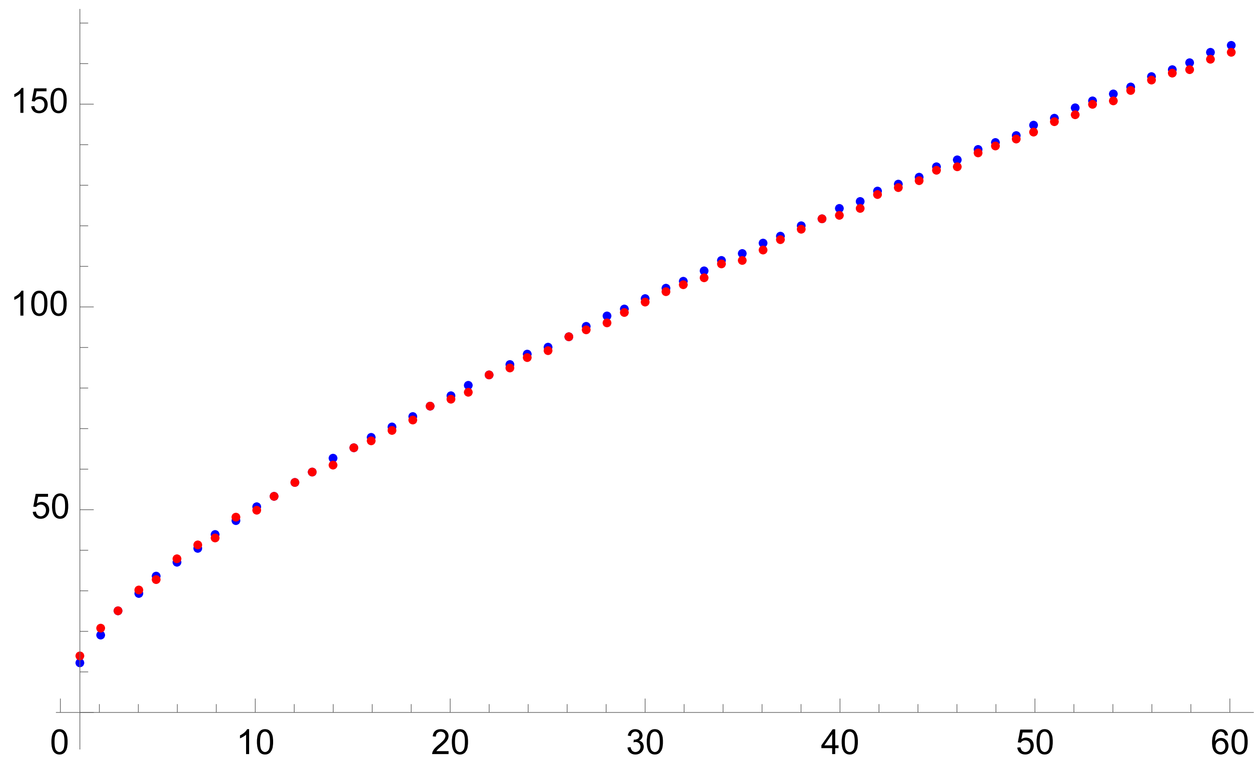
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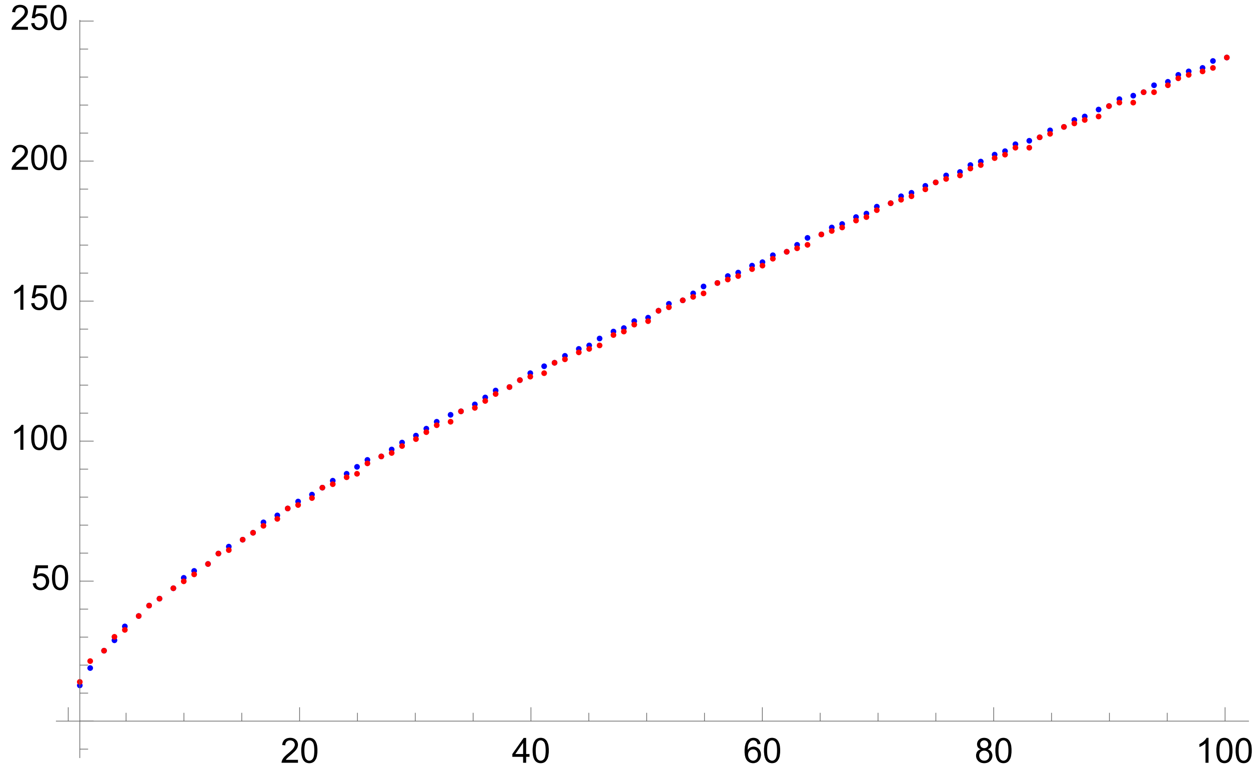
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80
100

Outlook

- Combine infrared with ultraviolet.

- Find the analogue of the prolate wave operator in the semilocal case.

- Understand the underlying geometry from Dirac operator.

Geometry= spectral triple

The metric associated to the spectral triple is

$$ds^2 = -\frac{1}{4}dx^2/(x^2 - \lambda^2) = \frac{1}{\alpha(x)}dx^2$$

Geometry is compactification of $2D$-Lorentzian with periodic time t

$$ds^2 = -\alpha(x)dt^2 + \frac{1}{\alpha(x)}dx^2$$

which after changing coordinates to $v = t - t(x)$ with

$$t(x) = \frac{1}{8\lambda} \log\left((\lambda + x)/(x - \lambda)\right)$$

becomes smooth

$$ds^2 = 4\left(x^2 - \lambda^2\right) dv^2 - 2dvdx$$