Properties characterising truth and satisfaction

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This is a joint work with Mateusz Łełyk.

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By UTB, the witness realises our type. □

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- What kind of definability we mean?

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The key fact:

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An extension $M \subseteq N$ is **conservative** iff for any A definable in N, the set

$$A \cap M$$

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There exists a recursively saturated model M of PA *which is rather classless, i.e., for any $X \subset M$ if X is piecewise coded, then X is definable.*

By "piecewise coded," we mean that for every c, the set $X \cap c$ is coded as a finite set in the sense of PA.

Let URS be a theory extending PA with the following axioms for any $n \in \omega$ and any recursive type $p = \phi_0, \phi_1, \ldots,$ where ϕ_i arithmetic:

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$$\forall \bar{y}\Big(\exists x \bigwedge_{i \leq n} \phi_i(x, \bar{y}) \rightarrow \bigwedge_{i \leq n} \phi_i(c_p(\bar{y}, \bar{y}))\Big).$$

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We verify that a model $M \models \mathrm{PA}$ expands to URS iff it is recursively saturated. In particular, if M is a rather classless recursively saturated model of PA, then it expands to URS but not to UTB.

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$$\neg\forall t_1 \ldots t_n \left(T_j\phi_i(t_1, \ldots, t_n) \equiv \phi_i(\mathrm{val}(t_1), \ldots, \mathrm{val}(t_n))\right) \longrightarrow$$
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In this model, $T_0, T_1, \ldots, T_N, T_\omega$ are all definable, so ϕ defines an arithmetical set, so it cannot define a predicate satisfying UTB. □

In particular, we see that th definability of UTB given by Kossak's result cannot be fully uniform.

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In particular, we see that th definability of UTB given by Kossak's result cannot be fully uniform.

Theorem

Suppose that U is a theory in a countable language extending PA. *Assume that for any $M \models U$, the arithmetical reduct of U is recursively saturated. Then in every model $M \models U$, we can define (with parametres) a set T such that $(M, T) \models \mathrm{UTB}^-$.*

Let U be as in the assumptions. We inductively define a sequence of sets of formulae A_α and a sequence of theories U_α in a language L_U with an extra constant a.

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$$U_\alpha \vdash \Big(\psi(a) \rightarrow \exists y \phi(a, y)\Big) \wedge \forall y \Big(\phi(a, y) \rightarrow \tau(a, y)\Big).$$

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Recall that $\tau(a, y)$ is the type: "y codes the part of elementary diagram with terms up to a." It makes sense to write a type in such a context. Let

$$U_{\alpha+1} := U_\alpha \cup \{\neg\psi \ \mid \ \psi \in A_\alpha\}.$$

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If U_α is consistent, then $A_\alpha \nsubseteq U_\alpha$.

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Lemma

For each α, if U_α is consistent, then $U_{\alpha+1} \supsetneq U_\alpha$.

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- If, on the other hand, U is not complete, then the dependence on the theory of the model is not uniform.
- The formulae defining the UTB^- predicate in general do not have some bounded complexity.
- A predicate provably satisfying UTB^- need not be definable in U, even if we assume that the language is finite.

Thank you for your attention!